

SUPERTROPICAL LINEAR ALGEBRA

ZUR IZHAKIAN, MANFRED KNEBUSCH, AND LOUIS ROWEN

ABSTRACT. The objective of this paper is to lay out the algebraic theory of supertropical vector spaces and linear algebra, utilizing the key antisymmetric relation of “ghost surpasses.” Special attention is paid to the various notions of “base,” which include d-base and s-base, and these are compared to other treatments in the tropical theory. Whereas the number of elements in a d-base may vary according to the d-base, it is shown that when an s-base exists, it is unique up to permutation and multiplication by scalars, and can be identified with a set of “critical” elements. Linear functionals and the dual space are also studied, leading to supertropical bilinear forms and a supertropical version of the Gram matrix, including its connection to linear dependence, as well as a supertropical version of a theorem of Artin.

1. INTRODUCTION

The objective of this paper is to lay out an algebraic theory for linear algebra in tropical mathematics. Extending the max-plus algebra to the supertropical algebra of [8] (which was designed as an algebraic foundation for tropical geometry), we obtain a theory paralleling the classical structure theory of commutative algebras.

Although there already is an extensive literature on tropical linear algebra over the max-plus algebra, including linear dependence [2] and matrix rank [1], the emphasis often is combinatoric or geometric. The traditional approach in semiring theory is to divide the determinant into a positive and negative part (since -1 need not exist in the semiring), cf. [15]. Whereas this approach provides many basic important properties of matrices, such as a general method given in [2] to transfer identities from ring theory to semiring theory, the reliance on combinatorics also leads to competing (and different) definitions. For example, in [1], five different definitions of matrix rank are given: The row rank, the Barvinok (Shein) rank, the strong rank, the Gondran-Minoux rank, the symmetrized rank, and the Kapranov rank.

The structure theory of supertropical semirings tends to unify these notions, giving a single formula for the determinant, from which we can define a nonsingular matrix; in this approach, the row rank, column rank, and strong rank all coincide. This makes it easier to proceed with a traditional algebraic development. Explicitly, properties of matrices were studied in [9], [10], [11], and [12], where the main theme is to replace the max-plus algebra by a cover, called the **supertropical semiring**, which permits one to formulate stronger results which are amenable to proofs more in line with classical matrix theory. Recall that the underlying supertropical structure is a semiring (without zero), R , with a designated semiring ideal $\mathcal{G} \supseteq mR$ for all m , where mR denotes $a + \cdots + a$ repeated m times; the algebraic significance is obtained by interpreting \mathcal{G} as “ghost elements”, elements which collectively are treated analogously to a zero element. When convenient, one assumes that R contains a zero element 0_R , which

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can be formally adjoined. Thus, we introduce the fundamental relation $a \models_{\text{gs}} b$ when a equals b plus a ghost element (which could be 0_R).

We recall that the **tropical determinant** of an $n \times n$ matrix $A = (a_{i,j})$ in $M_n(R)$ is really the permanent, which we denote as

$$|A| = \sum_{\pi \in S_n} a_{\pi(1),1} \cdots a_{\pi(n),n}.$$

Although the equation $|AB| = |A||B|$ fails over the max-plus algebra, the relation $|AB| \models_{\text{gs}} |A||B|$ holds over a supertropical semiring, [9, Theorem 3.5], and any matrix satisfies its characteristic polynomial in the sense of [9, Theorem 5.2]. The roots of this polynomial are precisely the supertropical eigenvalues.

Our main objective here is to initiate a formal theory of supertropical vector spaces and their bases, over semirings with ghosts, and in particular over supertropical semifields.

Our method is to rely as far as possible on the structure theory. While this theory parallels the classical theory of linear algebra, several key differences do emerge. At the outset, one major difference is that there are two different kinds of bases. First, one can take a maximal (tropically) independent set, which we call a **d-base**, called a “basis” in [14, Definition 5.2.4]. This has considerable geometric significance, intuitively providing a notion of rank (although, by an example in [14], the rank might vary according to the choice of d-base). As one might expect from [10], any dependence among vectors can be enlarged to an (often unique) **saturated dependence**, which is maximal in a certain sense; cf. Theorem 4.18. This leads to a delicate analysis of rank of a subspace, especially since it turns out that the number of elements in different d-bases may differ.

Alternatively, one can consider sets that (tropically) span the subspace; an **s-base** is a minimal such set when it exists. Such sets are used in generating convex spaces, as studied in [?]. Not every d-base is an s-base. In fact, the number of elements of an s-base might necessarily be larger than the number of elements of a d-base. Surprisingly, an s-base is unique up to scalar multiples, and can be characterized in terms of **critical** elements, which intuitively are elements that cannot be decomposed into sums of other elements. On the other hand, the d-bases can be quite varied, and lead us to interesting subspaces that they span, which we call **thick**.

We also consider linear transformations in this context, in which the equality $\varphi(v+w) = \varphi(v) + \varphi(w)$ is replaced by the ghost surpassing relation $\varphi(v+w) \models_{\text{gs}} \varphi(v) + \varphi(w)$. Linear transformations lead us to the notion of the **dual space**. The dual space depends on the choice \mathcal{B} of d-base, but there is a natural “dual s-base” of the dual space of \mathcal{B} , of the same rank (Theorem 6.20).

In the last section we introduce supertropical bilinear forms, in order to study “ghost orthogonality” between vectors. One calls two vectors v and w **g-orthogonal** with respect to a supertropical bilinear form $\langle \cdot, \cdot \rangle$ when $\langle v, w \rangle$ is a ghost. We construct the **Gram matrix** and prove the connection between tropical dependence of vectors in a nondegenerate space and the singularity of this matrix (Theorem 7.10). Finally, we prove (Theorem 7.20) a variant of Artin’s Theorem: When the g-orthogonality relation is symmetric, the supertropical bilinear form is “supertropically symmetric.”

Since the exposition [14] is an excellent source of fundamental results and examples, we use it as a general reference for the “standard” tropical theory and compare several of our definitions with the definitions given there.

2. SUPERTROPICAL STRUCTURES

2.1. Semirings without zero. A **semiring without zero**, which we notate as semiring^\dagger , is a structure $(\mathbb{R}, +, \cdot, 1_R)$ such that $(\mathbb{R}, \cdot, 1_R)$ is a monoid and $(\mathbb{R}, +)$ is a commutative semigroup, with distributivity of multiplication over addition on both sides. (In other words, a semiring^\dagger does not necessarily have the zero element 0 , but any semiring can also be considered as a semiring^\dagger .)

The reason one does not initially require a zero element is twofold: On the one hand, in contrast to ring theory, the zero element plays at best a marginal role in semirings, because of the lack of additive inverses, and often gets in the way, requiring special treatment in definitions and propositions; on the other hand, in our main example, the max-plus algebra of \mathbb{R} , the zero element does not exist in \mathbb{R} but is adjoined formally (as $-\infty$), and often gets in the way (for example, when one wants to evaluate Laurent

polynomials.) At any rate, given a semiring[†] R^\dagger , we can formally adjoin the element 0 to obtain the semiring $R := R^\dagger \cup \{0\}$, where we stipulate for all $a \in R$:

$$0 + a = a + 0 = a; \quad 0a = a0 = 0.$$

A **semiring[†] with ghosts** is a triple $(R^\dagger, \mathcal{G}, \nu)$, where R^\dagger is a semiring[†] and \mathcal{G} is a semiring[†] ideal, called the **ghost ideal**, together with an idempotent map

$$\nu : R^\dagger \longrightarrow \mathcal{G}$$

called the **ghost map on R^\dagger** , given by

$$\nu(a) = a + a.$$

(We require that ν preserves multiplication as well as addition.) We write a^ν for $\nu(a)$. Thus,

$$e := 1_{R^\dagger}^\nu$$

is both a multiplicative and additive idempotent of R^\dagger , which plays a key role since $\nu(R^\dagger) = eR^\dagger$.

A **supertropical semiring[†]** has the extra properties:

- (a) $a + b = a^\nu$ if $a^\nu = b^\nu$;
 - (b) $a + b \in \{a, b\}$, $\forall a, b \in R^\dagger$ s.t. $a^\nu \neq b^\nu$.
- (Equivalently, \mathcal{G} is ordered, via $a^\nu \leq b^\nu$ iff $a^\nu + b^\nu = b^\nu$.)

We write $a >_\nu b$ if $a^\nu > b^\nu$; we stipulate that a and b are ν -**matched**, written $a \cong_\nu b$, if $a^\nu = b^\nu$. We say that a **dominates** b if $a >_\nu b$.

Recall that any commutative supertropical semiring satisfies the **Frobenius formula** from [8, Remark 1.1]:

$$(a + b)^m = a^m + b^m \tag{2.1}$$

for any $m \in \mathbb{N}^+$.

A **supertropical domain[†]** [8] is a supertropical semiring[†] R^\dagger for which

$$\mathcal{T} := R^\dagger \setminus \mathcal{G}$$

is a multiplicative monoid, such that the map $\nu|_{\mathcal{T}} : \mathcal{T} \rightarrow \mathcal{G}$ (defined as the restriction from ν to \mathcal{T}) is onto. \mathcal{T} is called the set of **tangible elements** of R^\dagger . A **supertropical semifield[†]** is a supertropical domain[†] $(R^\dagger, \mathcal{G}, \nu)$ in which \mathcal{T} is a group. Thus, \mathcal{G} is also a group.

We have the analogous definitions when we adjoin the element 0_R to the semiring[†] R^\dagger to obtain the **semiring with ghosts R** . Thus, we write

$$R := R^\dagger \cup \{0_R\} = (R, \mathcal{G}_0, \nu),$$

where $\mathcal{G}_0 := \mathcal{G} \cup \{0_R\}$ is a semiring ideal, called the **ghost ideal**, and the ghost map $\nu : R \rightarrow \mathcal{G}_0$ satisfies $\nu(0_R) = 0_R$. Conversely, given a semiring with ghosts (R, \mathcal{G}_0, ν) , we can take $R^\dagger = R \setminus \{0_R\}$ and $\mathcal{G} = \mathcal{G}_0 \setminus \{0_R\}$ and define the semiring[†] with ghosts $(R^\dagger, \mathcal{G}, \nu)$. Thus, the theories with or without 0_R are basically the same.

In this spirit, we say that R is a **supertropical semiring** when R^\dagger is a supertropical semiring[†], and say that R is a **supertropical domain** when R^\dagger is a supertropical domain[†]; i.e., $\mathcal{T} = R \setminus \mathcal{G}_0$ is the monoid of tangible elements. (We write \mathcal{T}_0 for $\mathcal{T} \cup \{0_R\}$ in the supertropical domain R .) Likewise, a **supertropical semifield** is a supertropical domain (R, \mathcal{G}_0, ν) in which \mathcal{T} is a group.

Intuitively, the tangible elements correspond in some sense to the original max-plus algebra, although here $a + a = a^\nu$ instead of $a + a = a$. Our motivating example of supertropical semifield, used as the primary example throughout [8] as well as in this paper, is the extended tropical semiring [5]

$$\mathbb{T} := D(\mathbb{R}) := (\mathbb{R} \cup \mathbb{R}^\nu \cup \{-\infty\}, \mathbb{R}^\nu \cup \{-\infty\}, 1_{\mathbb{R}}),$$

the most familiar example of a supertropical semifield whose operations are induced by the standard operations \max and $+$ over the real numbers; we call this **logarithmic notation**, since the zero element $0_{\mathbb{T}}$ is $-\infty$ and the unit element $1_{\mathbb{T}}$ is 0 .

The supertropical domain, and in particular the supertropical semifield, seem to play a basic role in supertropical algebra parallel to the role of the field in classical algebra. In [8] a reduction is given from supertropical domains to supertropical semifields. Accordingly, one is led to study linear algebra over supertropical semifields.

Occasionally, we also want to pass back from \mathcal{G} to \mathcal{T} . Abusing notation slightly, we pick a representative in \mathcal{T} for each class in the image of $\hat{\nu}$, thereby getting a function

$$\hat{\nu} : R^\dagger \rightarrow \mathcal{T}$$

by putting $\hat{\nu}|_{\mathcal{T}} = 1_{\mathcal{T}}$; also, by definition, $\nu \circ (\hat{\nu}|_{\mathcal{G}}) = 1_{\mathcal{G}}$. In this case, we also write \hat{a} for $\hat{\nu}(a)$, but when the notation becomes cumbersome, we still use the $\hat{\nu}$ notation.

Here are two reductions to the case that $\nu_{\mathcal{T}}$ is 1:1.

Remark 2.1. We define an equivalence on R via $a \equiv b$ when either $a = b$ or $a, b \in \mathcal{T}$ with $a \cong_{\nu} b$. In other words, two tangible elements are equivalent iff they are ν -matched. Then we could define the supertropical domain[†]

$$\overline{R} := R^\dagger / \equiv$$

to be $(\mathcal{T} / \equiv) \cup \mathcal{G}$. The ghost map ν defines a 1:1 function from the equivalence classes of \mathcal{T} to \mathcal{G} .

Remark 2.2. In [10, Proposition 1.6] we see that $\hat{\nu}$ can be chosen to be multiplicative on \mathcal{G} . When \mathcal{G} is a multiplicative group, define

$$\tilde{T} := \hat{\nu}(\mathcal{G}) = \{a \in \mathcal{T} : \widehat{a^\nu} = a\},$$

and

$$R' := \tilde{T} \cup \mathcal{G},$$

and let ν' be the restriction of ν to R' . Then (R', \mathcal{G}, ν') is a supertropical domain[†], whose tangible elements are \tilde{T} , and $\nu'|_{\tilde{T}} : \tilde{T} \rightarrow \mathcal{G}$ is 1:1.

Remark 2.3. When $\nu_{\mathcal{T}}$ is not 1:1, it is convenient to define

$$\mathcal{T}_e := \{a \in \mathcal{T} : a \cong_{\nu} 1_R\}.$$

Note that \mathcal{T}_e is a submonoid of \mathcal{T} , and in fact $\mathcal{T}_e \cup \{e\}$ is a supertropical domain[†] contained in R .

To clarify our exposition, most of the examples in this paper are presented for the extended tropical semiring $D(\mathbb{R})$.

2.2. The “ghost surpass” and “ghost dependence” relations. We consider the semiring with ghosts (R, \mathcal{G}_0, ν) .

Definition 2.4. We say b is **ghost dependent** on a , written $b \underset{gd}{\Upsilon} a$, if $a + b \in \mathcal{G}_0$.

In particular, $a \cong_{\nu} b$ implies that $a \underset{gd}{\Upsilon} b$.

Note that the ghost dependence relation is symmetric, but not transitive, since $1 \underset{gd}{\Upsilon} 3^\nu$ and $3^\nu \underset{gd}{\Upsilon} 2$, although 1 and 2 are not ghost dependent. The following antisymmetric and transitive relation is a key to much of the theory.

Definition 2.5. We define the **ghost surpasses** relation $\underset{gs}{\models}$ on R , by

$$a \underset{gs}{\models} b \quad \text{iff} \quad a = b + c \quad \text{for some} \quad c \in \mathcal{G}_0.$$

In this notation, by writing $a \underset{gs}{\models} 0_R$ we mean $a \in \mathcal{G}_0$. This restricts to the **ghost surpasses** relation

on R^\dagger , by

$$a \underset{gs}{\models} b \quad \text{iff} \quad a = b \quad \text{or} \quad a = b + c \quad \text{for some} \quad c \in \mathcal{G}.$$

Remark 2.6. The following are equivalent:

- (1) $a \underset{gd}{\Upsilon} 0_R$;
- (2) $a \in \mathcal{G}_0$;
- (3) $a \underset{gs}{\models} 0_R$.

We quote some easy properties of $\underset{gs}{\models}$ from [10]:

Remark 2.7.

- (i) ([10, Remark 1.2]) When a is tangible, $a \models_{\text{gs}} b$ implies that $a = b$. In particular, tangible elements are comparable under \models_{gs} iff they are equal. In this way, the relation \models_{gs} generalizes equality.
- (ii) $a \models_{\text{gs}} b$ iff $a = b$ or a is a ghost $\geq_\nu b$. In particular, if $a \models_{\text{gs}} b$ then $a \geq_\nu b$; if $a \models_{\text{gs}} b$ for $b \in \mathcal{G}_0$, then $a \in \mathcal{G}_0$.
- (iii) ([10, Lemma 1.5]) \models_{gs} is an antisymmetric partial order on R .
- (iv) If $a \models_{\text{gs}} b$, then $a \curlyvee_{\text{gd}} b$.

Lemma 2.8. Generalizing Remark 2.7(i), for R a supertropical domain[†], an element $a \in R$ is tangible iff the following condition holds:

$a \models_{\text{gs}} b$ implies $b = \alpha a$ for some $\alpha \in \mathcal{T}_e$.

Proof. (\Rightarrow) is by Remark 2.7(i). Conversely, suppose a is not tangible; i.e. $a \in \mathcal{G}$, so $a = a^\nu$. Then $a \models_{\text{gs}} \hat{a}$, where $\hat{a} \in \mathcal{T}$ and $(\hat{a})^\nu = a$. The condition implies $\hat{a} = \alpha a$ for some $\alpha \in \mathcal{T}_e$, which is impossible since $\alpha a \in \mathcal{G}$. \square

This leads us later to a good abstract criterion for tangibility. Also, conversely to Remark 2.7(iv), we have

Lemma 2.9.

- (i) If $a \curlyvee_{\text{gd}} b$ with $b \in \mathcal{T}$, then either $a \equiv b$ or $a \models_{\text{gs}} b$.
- (ii) If $a \curlyvee_{\text{gd}} b$ with $a \geq_\nu b$, then either $a \equiv b$ or $a \models_{\text{gs}} b$.

Proof. (For both parts) If $a \in \mathcal{G}$ with $a \geq_\nu b$, then $a = b + a$. So we are done unless $a \in \mathcal{T}$, which implies $a \cong_\nu b$, and thus $a = b$, since $\nu|_{\mathcal{T}}$ is assumed to be 1:1. \square

2.3. Vector spaces with ghosts. Modules over semirings (often called “semimodules” in the literature [16], or sometimes “cones”) are defined just as modules over rings, except that now the additive structure is that of a semigroup instead of a group. (Note that subtraction does not enter into the other axioms of a module over a ring.)

Definition 2.10. Suppose R is a semiring. An R -**module** V is a semigroup $(V, +, \mathbb{0}_V)$ together with scalar multiplication $R \times V \rightarrow V$ satisfying the following properties for all $r_i \in R$ and $v, w \in V$:

- (1) $r(v + w) = rv + rw$;
- (2) $(r_1 + r_2)v = r_1v + r_2v$;
- (3) $(r_1r_2)v = r_1(r_2v)$;
- (4) $\mathbb{1}_R v = v$;
- (5) $r\mathbb{0}_V = \mathbb{0}_V$;
- (6) $\mathbb{0}_R v = \mathbb{0}_V$.

Note 2.11. One could also define module over a semiring[†], by deleting Axiom (6). In the other direction, any module V over a semiring[†] R^\dagger becomes an R -module when we formally define $\mathbb{0}_R v = \mathbb{0}_V$ for each $v \in V$.

The reason we prefer the terminology “module” is that this definition of module over a semiring R coincides with the usual definition of module when R is a ring, since $-v = (-\mathbb{1}_R)v$. In case the underlying semiring is a semiring with ghosts, V has the distinguished submodule eV , as well as the ghost map $\nu : V \rightarrow eV$, given by

$$\nu(v) := v + v = (\mathbb{1}_R + \mathbb{1}_R)v = ev \in eV.$$

Lemma 2.12. *Any R -module V over a semiring with ghosts R satisfies the following properties for all $r \in R, v \in V$:*

- (1) $(rv)^\nu = rv^\nu = r^\nu v$;
- (2) $(v + w)^\nu = v^\nu + w^\nu$.

Proof. (1) $(rv)^\nu = e(rv) = (er)v = (re)v = r(ev) = rv^\nu$.

(2) $(v + w)^\nu = e(v + w) = ev + ew = v^\nu + w^\nu$. \square

In order to obtain a stronger version of supertropicality we introduce the following definition:

Definition 2.13. *Suppose $R = (R, \mathcal{G}_0, \nu)$ is a semiring with ghosts. An R -module with ghosts (V, \mathcal{H}_0) is comprised of an R -module V and an R -submodule $\mathcal{H}_0 \supseteq eV$ satisfying the axiom:*

$$v^\nu = w^\nu \quad \text{implies} \quad v + w = v^\nu, \quad \forall v, w \in V.$$

We call \mathcal{H}_0 the **ghost submodule** of V , and ν is called the **ghost map** on V .

We define the map $\nu : V \rightarrow \mathcal{H}_0$, given by $\nu(v) := v + v = ev$, and write v^ν for $\nu(v)$.

The choice of the ghost submodule can be significant. (Note that v^ν could differ from v even when $v \in \mathcal{H}_0$.) The **standard ghost submodule** of V is defined as eV . Any module over a supertropical semiring can be viewed as a module with ghosts with respect to the standard ghost submodule eV ; in this case, we suppress \mathcal{H}_0 in the notation.

Definition 2.14. *An R -submodule with ghosts of (V, \mathcal{H}_0) is a submodule W of V , endowed with the ghost submodule $W \cap \mathcal{H}_0$, whose ghost map is the restriction of ν to W .*

When R is a supertropical semifield, (V, \mathcal{H}_0) is called a **(supertropical) vector space** over R , or vector space, for short. We focus on vector spaces in this paper, and call their elements **vectors**. A more general investigation of modules with ghosts is given in [7]. Our main example of a vector space in this paper, as well as in [9], is $R^{(n)} = (R^{(n)}, \mathcal{G}_0^{(n)})$, whose ghost map acts as ν on each component. The zero element $\mathbf{0}$ of $R^{(n)}$ is $(\mathbf{0}_R, \dots, \mathbf{0}_R)$. The tangible vectors of $R^{(n)}$ are those (v_1, \dots, v_n) such that each $v_i \in \mathcal{T}_0$.

As with semirings with ghosts, we define the **ghost surpassing relation** \models_{gs} for vectors $v, w \in V$ by:

$$v \models_{\text{gs}} w \quad \text{if} \quad v = w + u \text{ for some } u \in \mathcal{H}_0.$$

We say that two vectors $v, w \in V$ are ν -**matched**, written $v \cong_\nu w$, if $v^\nu = w^\nu$. Likewise, we write $v \geq_\nu w$ if $v^\nu = w^\nu + x^\nu$ for some $x^\nu \in \mathcal{H}_0$.

Example 2.15.

$$(v_1, \dots, v_n) \geq_\nu (w_1, \dots, w_n) \tag{2.2}$$

in $R^{(n)}$ iff $v_i \geq_\nu w_i$ for each $1 \leq i \leq n$.

Also, for elements v, w in a module with ghosts, we define

$$v \Upsilon_{\text{gd}} w \quad \text{if} \quad v + w \in \mathcal{H}_0.$$

Remark 2.16.

- (i) If $v \models_{\text{gs}} w$, then $v + w \in \mathcal{H}_0$, i.e., $v \Upsilon_{\text{gd}} w$.
- (ii) If $v_i \models_{\text{gs}} w$ for $i = 1, 2$, then $v_1 + v_2 \models_{\text{gs}} w$.

Lemma 2.17. *Any module with ghosts (V, \mathcal{H}_0) satisfies the following property, for all $v, w \in V, h \in eV$:*

$$v = w + h \Rightarrow v + h = v,$$

Proof. $v = w + h = w + h + h = v + h$. \square

Proposition 2.18. *Any module with ghosts (V, \mathcal{H}_0) satisfies the following property, for all $v, w \in V, h_1, h_2 \in \mathcal{H}_0$:*

$$v + h_1 + h_2 = v \Rightarrow v + h_2 = v,$$

Proof. $v = v + h_1 + h_2 = (v + h_1 + h_2) + (h_1 + h_2) = v + eh_1 + eh_2$. Take $w = v + eh_1$ and $h = eh_2$ in the lemma to get $v = v + eh_2 = (v + h_1 + h_2) + h_2 = v + h_2$. \square

Corollary 2.19. *The ghost surpassing relation is always antisymmetric.*

Motivated by Lemma 2.8, we have an abstract definition of tangibility for any vector space over a supertropical semifield (which is used more generally in [7] for any module over a supertropical semiring[†]):

Definition 2.20. *The **almost tangible vectors** of V are those elements $v \in V$ for which $v \models_{\text{gs}} w$ implies $w \in \mathcal{T}_e v$, $\forall w \in V$.*

Remark 2.21. *A nonzero ghost vector v cannot be almost tangible, for we always have*

$$v = \left(\mathbb{1}_R + \frac{\mathbb{1}_R}{2} \right) v = v + \frac{\mathbb{1}_R}{2} v \not\models_{\text{gs}} \frac{\mathbb{1}_R}{2} v.$$

Example 2.22. *Clearly, almost tangible vectors in $R^{(n)}$ are tangible.*

On the other hand, in logarithmic notation, taking $R = D(\mathbb{R})$, if V is the submodule of $R^{(2)}$ spanned by the vectors $v_1 = (1, 1^\nu)$ and $v_2 = (0, 1)$, then one sees without difficulty that v_1 is almost tangible in V , although not tangible in $R^{(2)}$.

In fact, a submodule of $R^{(n)}$ need not have any tangible vectors at all, as exemplified by the submodule $R(1, 1^\nu)$ of $R^{(2)}$.

Example 2.23. *For vectors $v = (v_1, \dots, v_n)$ and $w = (w_1, \dots, w_n)$ in $R^{(n)}$, $v \models_{\text{gs}} w$ iff $v_i \models_{\text{gs}} w_i$ for all $i = 1, \dots, n$. Thus, checking components, we see that the ghost surpassing relation for vectors of $R^{(n)}$ is antisymmetric.*

Here is another useful property of vectors in $R^{(n)}$.

Lemma 2.24. *If $v \models_{\text{gs}} \sum_{i=1}^{\ell} \alpha_i w_i$ and $v \models_{\text{gs}} \sum_{i=1}^{\ell} \alpha'_i w_i$ in $R^{(n)}$, then $v \models_{\text{gs}} \sum_{i=1}^{\ell} (\widehat{\alpha_i + \alpha'_i}) w_i$.*

Proof. Checking components, we may assume that $n = 1$. But then the assertion is immediate. \square

3. BACKGROUND FROM MATRICES

Any set $S = \{v_1, \dots, v_m\}$ of m row vectors in $R^{(n)}$ corresponds to an $m \times n$ matrix $A(S)$, whose m rows are the vectors of S . We call $A(S)$ the **matrix** of S .

We defined $|A|$ in the introduction. We say that the matrix A is **nonsingular** if $|A|$ is tangible (and thus quasi-invertible when R is a supertropical semifield [9]); otherwise, $|A| \in \mathcal{G}_0$ (i.e., $|A| \models_{\text{gs}} 0_F$ by

Remark 2.6) and we say that A is **singular**. In [9], we also defined vectors in

$$R^{(n)}$$

to be **tropically independent** if no linear combination with tangible coefficients is in \mathcal{H}_0 . By [12, Theorem 3.4], when R is a supertropical domain, $A(S)$ has m tropically independent rows iff $A(S)$ has a nonsingular $m \times m$ submatrix. Thus, it is natural to try to understand linear algebra in terms of the supertropical matrix theory of [9, 10].

Although it was shown in [9] that the product of nonsingular matrices could be singular, we do have the consolation that the product of nonsingular matrices cannot be ghost, cf. Theorem 3.4 below.

Recall that a **quasi-identity** matrix is a nonsingular, multiplicatively idempotent matrix ghost-surpassing the identity matrix. Suppose $A = (a_{i,j})$, with $|A|$ invertible in R . In [10, Theorem 2.8] one defines the matrix

$$A^\nabla := \frac{\mathbb{1}_R}{|A|} \text{adj}(A), \tag{3.1}$$

and obtains the quasi-identity matrices

$$I_A = AA^\nabla; \quad I'_A = A^\nabla A. \tag{3.2}$$

3.1. Annihilators of matrices.

Definition 3.1. A vector $v \in R^{(n)}$ (written as a column) ***g-annihilates*** an $m \times n$ matrix A if $Av \models_{\text{gs}} 0_V$ in $R^{(n)}$. Define

$$\text{Ann}(A) = \left\{ v \in R^{(n)} : Av \models_{\text{gs}} 0_V \right\},$$

clearly a submodule of $R^{(n)}$.

Accordingly, $\mathcal{G}_0^{(n)} \subseteq \text{Ann}(A)$, for any $m \times n$ matrix A .

Remark 3.2.

- (i) The point of this definition is that the vector $v = (\beta_1, \dots, \beta_m)$ *g-annihilates* $(A(S))^t$, the transpose of the matrix of $S = \{w_1, \dots, w_m\}$, iff $\sum_{i=1}^m \beta_i w_i \models_{\text{gs}} 0_R$. Thus, tangible *g-annihilators* correspond to tropical dependence relations.
- (ii) A (nonzero) tangible vector cannot *g-annihilate* a nonsingular matrix, since the columns are tropically independent.

We can improve this result, to include vectors that are not necessarily tangible.

Lemma 3.3. The diagonal of the product $I_A I_B$ of quasi-identity matrices I_A, I_B cannot all be ghosts.

Proof. Otherwise, write $I_A = (a_{i,j})$ and $I_B = (b_{i,j})$. If the assertion is false, then for each i_t there is i_{t+1} such that $a_{i_t, i_{t+1}} b_{i_{t+1}, i_t} \geq_\nu 1_F$. Consider the digraph G of $I_A I_B$, cf. [9, §3.2]. By the pigeonhole principle, the path of vertices $i_1, i_2, i_3, \dots, i_{n+1}$ contains a cycle, say from i_s to i'_s . But the weight of any non-loop cycle in a quasi-identity has ν -value less than 1_R . (Otherwise, multiplying by the entries $a_{i,i}$ for all vertices i not in the cycle gives an extra summand $\geq e = 1_R^\nu$ for $|I_A|$, contrary to $|I_A| = 1_R$.) Hence

$$1_R \leq_\nu \prod_{k=s}^{s'-1} a_{i_k, i_{k+1}} b_{i_{k+1}, i_k} = \prod_{k=s}^{s'-1} a_{i_k, i_{k+1}} \prod_{k=s}^{s'-1} b_{i_{k+1}, i_k} <_\nu 1_R 1_R = 1_R,$$

a contradiction. □

Theorem 3.4. The product of two nonsingular $n \times n$ matrices cannot be in $M_n(\mathcal{G}_0)$.

Proof. If AB is ghost for A, B nonsingular, then in the notation of [9, Definition 4.6],

$$I_{A^\nabla} I_B = I'_A I_B = A^\nabla A B B^\nabla \in M_n(\mathcal{G}_0),$$

contradicting the lemma. □

On the other hand, examples were given in [9] in which the product of two nonsingular $n \times n$ matrices is singular. Here is a related example using quasi-identities:

Example 3.5. The matrices

$$A = \begin{pmatrix} 0 & 0^\nu \\ -\infty & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -\infty \\ 0^\nu & 0 \end{pmatrix}$$

over $D(\mathbb{R})$ are nonsingular, but $AB = \begin{pmatrix} 0^\nu & 0^\nu \\ 0^\nu & 0 \end{pmatrix}$ and $BA = \begin{pmatrix} 0 & 0^\nu \\ 0^\nu & 0^\nu \end{pmatrix}$ are singular.

4. TROPICAL DEPENDENCE

Throughout the remainder of this paper, $F = (F, \mathcal{G}_0, \nu)$ denotes a supertropical semifield.

Dependence plays a major role in module theory. For supertropical modules with ghosts, the familiar definition becomes degenerate. The following modification from [9], in which the role of zero is replaced by the ghost ideal, is more suitable for our purposes.

Definition 4.1. Suppose (V, \mathcal{H}_0) is a vector space over F . A family of elements $S = \{w_i : i \in I\} \subset V$ is **tropically dependent** if there exists a nonempty finite subset $I' \subset I$ and a family $\{\alpha_i : i \in I'\} \subset \mathcal{T}$, such that

$$\sum_{i \in I'} \alpha_i w_i \in \mathcal{H}_0. \quad (4.1)$$

Any such relation (4.1) is called a **tropical dependence** for S . A subset $S \subset V$ is called **tropically independent** if it is not tropically dependent.

Given an element $v \in V$, we say that v is **tropically dependent on** a family $S = \{w_i : i \in I\}$ if $S \cup \{v\}$ is tropically dependent, in which case we write $v \underset{gd}{\Upsilon} S$. (In particular, $v \underset{gd}{\Upsilon} \{v\}$.) A subset S' of V is **tropically dependent on** S if $v \underset{gd}{\Upsilon} S$ for each $v \in S'$.

An easy observation:

Remark 4.2. Suppose $S = \{w_i : i \in I\} \subset V$. For any given set $\{\alpha_i : i \in I\} \subset \mathcal{T}$ of tangible elements of R , the set S is tropically independent iff $\{\alpha_i w_i : i \in I\}$ is tropically independent.

Also recall that any $n + 1$ vectors of $R^{(n)}$ are tropically dependent, by [9, Corollary 6.7].

4.1. Tropical d-bases and rank.

Definition 4.3. A **d-base** (for dependence base) of a supertropical vector space V is a maximal set of tropically independent elements of V . The **rank** of a d-base \mathcal{B} , denoted $\text{rk}(\mathcal{B})$, is the number of elements of \mathcal{B} .

Our d-base corresponds to the “basis” in [14, Definition 5.2.4].

Proposition 4.4. Any subspace of $F^{(n)}$ is tropically dependent on any subset S of n tropically independent elements. All d-bases of $F^{(n)}$ have precisely n elements.

Proof. By [9, Theorem 6.6], the matrix A of S is nonsingular iff S is tropically independent, so in particular any d-base \mathcal{B} of $F^{(n)}$ must have at least n elements. On the other hand, any $n + 1$ vectors in $F^{(n)}$ are tropically dependent, by [9, Remark 1.1], so \mathcal{B} has precisely n elements. \square

This leads us to the following definition.

Definition 4.5. The **rank** of a supertropical vector space V is defined as:

$$\text{rk}(V) := \max \{ \text{rk}(\mathcal{B}) : \mathcal{B} \text{ is a d-base of } V \}.$$

We have just seen that $\text{rk}(F^{(n)}) = n$.

Corollary 4.6. If $V \subset F^{(n)}$, then $\text{rk}(V) \leq n$.

Proof. Any d-base of V is contained in a d-base of $F^{(n)}$, whose order must be that of the standard base (to be given in §5.1), which is n . \square

We might have liked $\text{rk}(V)$ to be independent of the choice of d-base of V , for any supertropical vector space V . This is proved in the classical theory of vector spaces by showing that dependence is transitive. However, transitivity fails in the supertropical theory, since we have the following sort of counterexample.

Example 4.7. In logarithmic notation, over $D(\mathbb{R})^{(3)}$, the vector $v = (0, 1, 3)$ is tropically dependent on $W = \{w_1, w_2\}$, where $w_1 = (1, 1, 2)$ and $w_2 = (1, 1, 3)$, since $v + w_1 + w_2 = (1^\nu, 1^\nu, 3^\nu)$. Furthermore, W is tropically dependent on $U = \{u_1, u_2\}$, where $u_1 = (1, 1, 0)$ and $u_2 = (-\infty, -\infty, 1)$, since

$$w_1 + u_1 + 1u_2 = (1^\nu, 1^\nu, 2^\nu), \quad w_2 + u_1 + 2u_2 = (1^\nu, 1^\nu, 3^\nu).$$

But v, u_1 , and u_2 are tropically independent, since the tropical determinant of the matrix whose rows are these vectors is $3 \in \mathcal{T}$.

In fact, different d-bases may contain different numbers of elements, even when tangible. An example is given in [14, Example 5.4.20], which is reproduced here with different entries.

Example 4.8. Consider the following vectors in $D(\mathbb{R})^{(3)}$:

$$v_1 = (5, 5, 0), \quad v_2 = (5, 5, 4), \quad v_3 = (0, 1, 4), \quad v_4 = (0, 2, 4).$$

Then v_1, v_2 , and v_3 are tropically dependent (since their sum $(5^\nu, 5^\nu, 4^\nu)$ is ghost) and likewise v_1, v_2 , and v_4 are tropically dependent. It follows that $\{v_1, v_2\}$ is a d -base for the supertropical vector space V spanned by v_1, v_2, v_3 , and v_4 . But v_2, v_3 , and v_4 are tropically independent since their determinant is 11, which is tangible; hence, $\{v_2, v_3, v_4\}$ is also a d -base of V .

We do have a consolation.

Lemma 4.9. If the vectors $v_1, \dots, v_k \in F^{(n)}$ are tropically independent and the vector v is tangible, then there are i_1, \dots, i_{k-1} in $\{1, \dots, k\}$ such that the vectors $v_{i_1}, \dots, v_{i_{k-1}}, v$ are tropically independent.

Proof. Let A be the $(k+1) \times n$ matrix whose rows are v_1, \dots, v_k, v , and let A_0 denote the $k \times n$ matrix of the first k rows v_1, \dots, v_k . By [12, Theorem 3.4], A_0 has a nonsingular $k \times k$ submatrix obtained by deleting $n - k$ columns; deleting these columns in A , we have reduced to the case that $n = k$; i.e., A is a $(k+1) \times k$ matrix. Now let $A'_0 = (a'_{j,i})$ denote the adjoint matrix of A_0 . We are done unless for each row $i \leq k$, the $k \times k$ submatrix of A obtained by deleting the i row is singular, which means that $\sum_{j=1}^k a'_{i,j} a_{k+1,j}$ is ghost. This means that the vector $(a_{k+1,1}, \dots, a_{k+1,k})$ g -annihilates the matrix A'_0 , which is nonsingular by [10, Theorem 4.9], an impossibility in view of [9, Corollary 6.6]. \square

Proposition 4.10. For any tropical subspace V of $F^{(n)}$ and any tangible $v \in V$, there is a tangible d -base of V containing v whose rank is that of V .

Proof. Take a tangible d -base of V of maximal rank, and apply the lemma. \square

Example 4.11. Failure of the analog of Proposition 4.10 for non-tangible vectors: Consider the supertropical vector space $W \subset D(\mathbb{R})^{(2)}$ spanned by $w_1 = (0, 1)$ and $w_2 = (0, 2)$. Then $v = (1, 3^\nu)$ comprises a d -base of W , consisting of only one element.

Proposition 4.12. If A is a matrix of rank m , its g -annihilator has a tangible tropically independent set of rank $\geq n - m$.

Proof. Take m tropically independent rows v_1, \dots, v_m of A , which we may assume are the first m rows of A . For any other row v_u of A ($m < u \leq n$), we have $\beta_{u,1}, \dots, \beta_{u,m} \in \mathcal{T}_0$ such that $v_u + \sum \beta_{i,j} v_i \in \mathcal{G}_0^{(n)}$. Letting B be the $(n-m) \times n$ matrix whose (i, j) entries are $\beta_{i,j}$ for $1 \leq i, j \leq m$, and for which $\beta_{i,j} = \delta_{i,j}$ (the Kronecker delta) for $m < j \leq n$, we see that B contains an $(n-m) \times (n-m)$ identity submatrix so has tangible rank $\geq n - m$, but BA is ghost. \square

Example 4.13. An example of a 3×3 matrix A over $D(\mathbb{R})$ of rank $m = 2$, all of whose entries are tangible, although $\text{rk}(\text{Ann}(A)) = 2 > 3 - 2$. Take

$$A = \begin{pmatrix} 4 & 4 & 0 \\ 4 & 4 & 1 \\ 4 & 4 & 2 \end{pmatrix}.$$

A is g -annihilated by the tropically independent vectors $v_1 = (1, 1, 0)^t$ and $v_2 = (1, 1, 1)^t$, since $Av_1 = Av_2 = (5, 5, 5)^t$.

Note that this kind of example requires $n \geq 3$, in view of Theorem 3.4.

4.2. Saturated dependence relations. Let us study tropical dependence relations in $R^{(n)}$ more closely. Example 5.8(ii) below shows that a tropical dependence of a vector v on an independent set $S = \{w_i : i \in I\}$ is not determined uniquely. Nevertheless, in this subsection we do get a “canonical” tropical dependence relation, which we call **saturated**. But first, in order for tropical dependence relations to be well-defined with respect to the ghost map $\nu : R \rightarrow \mathcal{G}_0$, we verify the following condition.

Lemma 4.14. Any submodule of $R^{(n)}$ (with the standard ghost submodule $\mathcal{H}_0 = \mathcal{G}_0^{(n)}$) satisfies the property that whenever $\alpha_i, \beta_i \in \mathcal{T}$ with $\alpha_i \cong_\nu \beta_i$,

$$\sum_i \alpha_i w_i \in \mathcal{H}_0 \quad \text{iff} \quad \sum_i \beta_i w_i \in \mathcal{H}_0. \quad (4.2)$$

Proof. The condition clearly passes to submodules, so it is enough to prove it for $R^{(n)}$, and thus, to check (4.2) on each component. We write $w_{i,j}$ for the j -component of w_i . Note that $\alpha_i w_{i,j} \cong_\nu \beta_i w_{i,j}$ for each i . There are two ways for $\sum_i \alpha_i w_{i,j} \in \mathcal{G}_0$:

- (1) Some $\alpha_{i'} w_{i',j}$ dominates $\sum \alpha_i w_{i,j}$ and is ghost, implying $w_{i',j} \in \mathcal{G}_0$, so

$$\sum \beta_i w_{i,j} = \beta_{i'} w_{i',j} = \alpha_{i'} w_{i',j} \in \mathcal{G}_0.$$

- (2) Two essential summands $\alpha_{i'} w_{i',j}$ and $\alpha_{i''} w_{i'',j}$ are ν -matched. But then

$$\begin{aligned} \sum \beta_i w_{i,j} &= \beta_{i'} w_{i',j} + \beta_{i''} w_{i'',j} = (\beta_{i'} w_{i',j})^\nu \\ &= (\alpha_{i'} w_{i',j})^\nu = \alpha_{i'} w_{i',j} + \alpha_{i''} w_{i'',j} = \sum \alpha_i w_{i,j} \in \mathcal{G}_0. \end{aligned}$$

□

We examine the tropical dependence

$$v \curlyvee_{gd} \sum_{i \in I} \alpha_i w_i, \quad (4.3)$$

Lemma 4.15. *Suppose $V = R^{(n)}$. If $v \curlyvee_{gd} \sum_{i \in I} \alpha_i w_i$ and $v \curlyvee_{gd} \sum_{i \in I} \beta_i w_i$, for $\alpha_i, \beta_i \in \mathcal{T}_0$, then taking $\gamma_i = \widehat{\alpha_i + \beta_i}$, we have*

$$v \curlyvee_{gd} \sum_{i \in I} \gamma_i w_i.$$

Proof. Checking each component in turn, we may assume that $V = R$. We proceed as in Lemma 4.14. Namely, $v \curlyvee_{gd} \sum_{i \in I} \alpha_i w_i$ (resp. $v \curlyvee_{gd} \sum_{i \in I} \beta_i w_i$) implies one of the following:

- (1) v and some term $\alpha_{i'} w_{i'}$ dominate (resp. v and $\beta_{i'} w_{i'}$ dominate), in which case $\gamma_{i'} = \alpha_{i'}$ (resp. $\gamma_{i'} = \beta_{i'}$).
- (2) $\alpha_{i'} w_{i'}$ and $\alpha_{i''} w_{i''}$ dominate, (resp. $\beta_{i'} w_{i'}$ and $\beta_{i''} w_{i''}$ dominate), in which case $\gamma_{i'} = \alpha_{i'}$ and $\gamma_{i''} = \alpha_{i''}$ (resp. $\gamma_{i'} = \beta_{i'}$ and $\gamma_{i''} = \beta_{i''}$).
- (3) Some ghost term $\alpha_{i'} w_{i'}$ (resp. $\beta_{i'} w_{i'}$) dominates, in which case $\gamma_{i'} = \alpha_{i'}$ (resp. $\gamma_{i'} = \beta_{i'}$).

□

Lemma 4.15 gives us a partial order on the coefficients of the tropical dependence relations of v on a set S , and motivates the following definition:

Definition 4.16. *We say that the **support** of a tropical dependence $\alpha v \curlyvee_{gd} \sum_{i \in I} \alpha_i w_i$ (where $\alpha \in \mathcal{T}$ and $\alpha_i \in \mathcal{T}_0$) is the set $\{i \in I : \alpha_i \neq 0_R\}$. A tropical dependence of minimal support is called **irredundant**.*

A tropical dependence of v on a tropically independent set S is called **saturated** if the coefficients α_i 's in Formula (4.3) are maximal possible with respect to \geq_ν , as defined in Equation (2.2); in other words, whenever $v + \sum_{i=1}^\ell \beta_i w_i \in \mathcal{G}_0^{(n)}$ with $\beta_i \in \mathcal{T}_0$, then each $\beta_i \leq_\nu \alpha_i$.

Remark 4.17. *If*

$$v \curlyvee_{gd} \sum_{i=1}^\ell \alpha_i w_i \quad (4.4)$$

is a saturated tropical dependence, then, for any $k \leq \ell$ and for $v' = v + \sum_{i=1}^k \alpha_i w_i$,

$$v' \curlyvee_{gd} \sum_{i=k+1}^\ell \alpha_i w_i \quad (4.5)$$

is also a saturated tropical dependence, since any ν -larger tropical dependence for (4.5) would yield the corresponding ν -larger tropical dependence for (4.4).

Theorem 4.18. *Suppose $V = F^{(n)}$, for a supertropical semifield $F = (F, \mathcal{G}, \nu)$. Any irredundant tropical dependence*

$$v \curlywedge_{gd} \sum_{i=1}^{\ell} \alpha_i w_i \quad (4.6)$$

can be increased to a unique (up to equivalence in the sense of Remark 2.1) saturated tropical dependence of v on $S = \{w_1, \dots, w_\ell\}$, having the same support.

Remark 4.19. *When the vector v is tangible, and S is a d -base, Theorem 4.18 is an immediate consequence of [10, Theorems 3.5 and 3.8], which shows that $Ax \models_{gs} v$ has the maximal tangible vector solution $x = \hat{\nu}(A^\nabla v)$ (where $A^\nabla = \frac{1}{|A|} \text{adj}(A)$), which in view of Lemma 2.9 is also a solution for $Ax \curlywedge_{gd} v$. Here we take A to be the matrix of S , which is nonsingular, and x to be the vector $(\alpha_1, \dots, \alpha_\ell)^t$.*

In general, $x = A^\nabla v$ is a solution for the matrix equation $Ax \models_{gs} v$, which, when v is written as a row, is $xA^t \models_{gs} v$. (In a sense, row form is more natural, since the matrix of S is obtained from the rows.) But this vector need not be tangible.

Here is a direct combinatoric proof of Theorem 4.18 that does not rely on matrix theory, and does not depend on the additional assumption of tangibility of S .

Proof. Uniqueness of a saturated tangible solution is obvious, since one could just take the sup of any two distinct saturated tropical dependences to get a contradiction. This also gives the motivation for proving existence. Write $v = (v_1, \dots, v_n)$. We start with some tropical dependence (4.6), which need not be saturated, with the aim of checking whether we can modify it until it is saturated. In principle, we could increase the ν -values of the coefficient α_i if at each component j of the vector $\alpha_i w_i$ the ν -value of v_j is not attained, and this is the main idea behind the proof. But increasing α_i still may not yield a saturated tropical dependence, since the coefficient may be allowed to increase further, so long as some other term in the tropical dependence also is adjusted so as to have a j -component of the same ν -value. Since these j -components are the most difficult to keep track of, we pay special attention to them. Write $w_{i,j}$ for the j -component of w_i .

We say that an index $j \leq n$ has **type 1** if v_j is not dominated by $\sum \alpha_j w_{i,j}$, which means that either v_j itself is ghost, or else precisely one w_i has $\alpha_i w_{i,j}$ matching v_j and this $w_{i,j} \in \mathcal{T}$.

We say that j has **type 2** for v if v_j is dominated by $\sum \alpha_j w_{i,j}$, which means that either there exists i such that $w_{i,j}$ is ghost and dominates v_j or there are i, i' such that both $\alpha_i w_{i,j}$ and $\alpha_{i'} w_{i',j}$ dominate v_j .

Note that increasing the coefficients α_i in a tropical dependence cannot change the type of an index j from type 2 to type 1. Also, at least one index must have type 1, since otherwise $\sum \alpha_i w_{i,j} \in \mathcal{G}_0^{(n)}$, contrary to the hypothesis that the w_i are tropically independent. We choose our tropical dependence such that the number of indices of type 1 is minimal. In this case, if $\alpha_i w_{i,j}$ ν -matches v_j for j of type 1, we cannot find a ν -greater tropical dependence in which α_i is increased, since this would force the tropical dependence to have an extra type 2 index. Thus, in this case we say w_i is **anchored** at j . Renumbering the vectors, we may assume that w_1, \dots, w_k are anchored at various indices, and replace v by $v' = v + \sum_{i=1}^k \alpha_i w_i$. Now we have a new tropical dependence $v' + \sum_{i=k+1}^{\ell} \alpha_i w_i \in \mathcal{G}_0^{(n)}$, which by induction on ℓ can be increased to a saturated tropical dependence

$$v' \curlywedge_{gd} \sum_{i=k+1}^{\ell} \alpha'_i w_i.$$

But then the tropical dependence

$$v \curlywedge_{gd} \sum_{i=1}^k \alpha_i w_i + \sum_{i=k+1}^{\ell} \alpha'_i w_i$$

is saturated, since w_1, \dots, w_k are anchored. □

Proposition 4.20. *If*

$$v \underset{gd}{\Upsilon} \sum_{i=1}^{\ell} \alpha_i w_i, \quad v' \underset{gd}{\Upsilon} \sum_{i=1}^{\ell} \alpha'_i w_i \quad (4.7)$$

are saturated tropical dependences, then

$$v + v' \underset{gd}{\Upsilon} \sum_{i=1}^{\ell} (\widehat{\alpha_i + \alpha'_i}) w_i \quad (4.8)$$

also is a saturated tropical dependence.

Proof. Again we have two proofs, the first using results from [10] in the case when v, v' are tangible and the matrix A of the w_i is nonsingular. In the first case, one just takes the solutions $x = \widehat{A^\nabla v}$ and $x' = \widehat{A^\nabla v'}$ for the vectors of the α_i and the α'_i , and then note that

$$\hat{v}(A^\nabla v + A^\nabla v') = \hat{v}(A^\nabla (v + v')).$$

For the general case, one needs to modify the second proof of Theorem 4.18 for the vector $v + v'$. Namely, consider the tropical dependence

$$v + v' \underset{gd}{\Upsilon} \sum_{i=1}^{\ell} \gamma_i w_i,$$

where $\gamma_i = (\widehat{\alpha_i + \alpha'_i})$. At least one index in this tropical dependence must have type 1 for $v + v'$, since otherwise the w_i are tropically dependent. We choose our tropical dependence such that the number of indices of type 1 is minimal. As before, if $\gamma_i w_{i,j}$ ν -matches v_j for j of type 1 we cannot find a larger tropical dependence in which γ_i is increased, so w_i is anchored at j . Again, we may assume that w_1, \dots, w_k are anchored at various indices, and replace $v + v'$ by

$$v'' = v + v' + \sum_{i=1}^k \gamma_i w_i.$$

But

$$v + \sum_{i=1}^k \alpha_i w_i \underset{gd}{\Upsilon} \sum_{i=k+1}^{\ell} \alpha_i w_i \quad \text{and} \quad v' + \sum_{i=1}^k \alpha'_i w_i \underset{gd}{\Upsilon} \sum_{i=k+1}^{\ell} \alpha'_i w_i$$

are saturated tropical dependences by Remark 4.17, so, by induction on ℓ ,

$$v'' \underset{gd}{\Upsilon} \sum_{i=k+1}^{\ell} \gamma_i w_i$$

is a saturated tropical dependence. But then the tropical dependence

$$v \underset{gd}{\Upsilon} \sum_{i=1}^k \gamma_i w_i + \sum_{i=k+1}^{\ell} \gamma_i w_i$$

is saturated. □

5. TROPICAL SPANNING

In this section, we continue to consider the fundamental question of what “base” should mean for supertropical vector spaces. The d-base (defined above) competes another notion to be obtained from \models . But at the moment we turn to the naive analog from the classical theory of linear algebra.

5.1. Classical bases.

Definition 5.1. A module V over a semiring R is **classically spanned** by a set $S = \{w_i : i \in I\}$ if every element of V can be written in the form

$$v = \sum_{i \in J} r_i w_i,$$

for $r_i \in R$ and some finite index set $J \subset I$.

A set $\mathcal{B} = \{b_1, \dots, b_n\} \subset V$ is a **classical base** of a module V over a semiring R , if every element of V can be written uniquely in the form $\sum_{i=1}^n r_i b_i$, for $r_i \in R$. In this case, we say that V is **classically free of rank n** .

For example, the **standard base** of $R^{(n)}$ is the classical base defined as

$$\varepsilon_1 = (\mathbb{1}_R, \mathbb{0}_R, \dots, \mathbb{0}_R), \quad \varepsilon_2 = (\mathbb{0}_R, \mathbb{1}_R, \mathbb{0}_R, \dots, \mathbb{0}_R), \quad \dots, \quad \varepsilon_n = (\mathbb{0}_R, \mathbb{0}_R, \dots, \mathbb{1}_R). \quad (5.1)$$

Proposition 5.2. If V is classically free of rank n , then V is isomorphic to $R^{(n)}$.

The proof is standard; taking a classical base b_1, \dots, b_n , one defines the isomorphism $R^{(n)} \rightarrow V$ by

$$(r_1, \dots, r_n) \mapsto \sum_{j=1}^n r_j b_j.$$

5.2. Tropical spanning.

Definition 5.3. A vector $v \in V$ is **tropically spanned** by a set $S = \{w_i : i \in I\} \subset V$ if there exists a nonempty finite subset $I' \subset I$ and a family $\{\alpha_i : i \in I'\} \subset \mathcal{T}$, such that

$$v \models_{\text{gs}} \sum_{i \in I'} \alpha_i w_i. \quad (5.2)$$

In this case, we write $v \models_{\text{gs}} S$.

A subset $S' \subseteq V$ is **tropically spanned** by S , written $S' \models_{\text{gs}} S$, if $v \models_{\text{gs}} S$ for each $v \in S'$.

Remark 5.4 (Transitivity for tropically spanning). If $V \models_{\text{gs}} W$ and $W \models_{\text{gs}} U$, then $V \models_{\text{gs}} U$.

Obviously, any set classically spanned by S is tropically spanned; surprisingly, the converse often holds.

Remark 5.5.

- (i) Any element tropically spanned by $S = \{w_i : i \in I\}$ is tropically dependent on S .
- (ii) If an almost tangible vector $v \in V$ is tropically spanned by a set $S \subset V$, then v is classically spanned by S .
- (iii) The assertion (ii) can fail for nontangible $v \in R^{(n)}$; take $S = \{(\mathbb{1}_R, \mathbb{1}_R)\} \subset R^{(2)}$, viewed as an R -module, then $(\mathbb{1}_R, \mathbb{1}_{R^\vee})$ is tropically spanned by S , but not classically spanned by S .
- (iv) If V has a classical spanning set \mathcal{B} of almost tangible vectors, and \mathcal{B} is tropically spanned by a set S , then V is classically spanned by S , by (ii) and transitivity. In particular, if $R^{(n)}$ is tropically spanned by a set S , then $R^{(n)}$ is classically spanned by S , since $R^{(n)}$ has the standard base.
- (v) Any element tropically spanned by S is also tropically dependent on S , but not conversely; for example $v = (\mathbb{1}_R, \mathbb{1}_R) \in R^{(2)}$ is tropically dependent on $S = \{(\mathbb{1}_R, \mathbb{1}_{R^\vee})\} \subset R^{(2)}$, viewed as R -module, but v is not tropically spanned by S . This leads to an interesting dichotomy to be studied shortly.

Thus, we see that almost tangible vectors already begin to play a special role in the theory of tropical dependence.

Remark 5.6. *Tropical spanning does not satisfy the assertion analogous to Lemma 4.15. For example, take*

$$\{w_1 = (1, 2), w_2 = (1, 3)\} \subset D(\mathbb{R})^{(2)}$$

and the vector $v = (1, 3^\nu)$, then $v \models_{\text{gs}} w_1$ and $v \models_{\text{gs}} w_2$, but $v \not\models_{\text{gs}} w_1 + w_2 = (1^\nu, 3)$.

Lemma 5.7. $W = \{v \in V : v \models_{\text{gs}} S\}$ is a subspace of V for any $S \subset V$.

Proof. If $v = \sum_{i \in I} \alpha_i w_i + y$ and $v' = \sum_{i \in I} \alpha'_i w_i + z$, where $\alpha_i, \alpha'_i \in \mathcal{T}$, $w_i \in S$ and $y, z \in \mathcal{H}_0$, then letting $J = \{i : \alpha_i \cong_\nu \alpha'_i\}$, we have, by bipotence,

$$v + v' = \sum_{i \notin J} \beta_i w_i + \sum_{i \in J} \alpha'_i w_i + (y + z) \models_{\text{gs}} \sum_{i \notin J} \beta_i w_i,$$

where $\beta_i \in \{\alpha_i, \alpha'_i\} \subset \mathcal{T}$. The other verifications are easier. \square

We call W (in Lemma 5.7) the subspace **tropically spanned** by S , and say that S is a **tropically spanning set** of W .

A supertropical vector space is **finitely spanned** if it has a finite tropically spanning set.

Example 5.8. Take $R = D(\mathbb{R})$, with logarithmic notation.

(i) The vectors

$$v_1 = (1, 0, 1), \quad v_2 = (1, 1, 0), \quad \text{and} \quad v_3 = (0, 1, 1)$$

are tropically dependent in $D(\mathbb{R})^{(3)}$, since their sum is $(1^\nu, 1^\nu, 1^\nu)$. None of these vectors is tropically spanned by the two other vectors.

(ii) Even when a vector is classically spanned by tropically independent vectors, the coefficients need not be unique. For example,

$$(4, 5) = 2(1, 1) + 2(2, 3) = 1(1, 1) + 2(2, 3).$$

The point of this example is that the first coefficient is sufficiently small so as not to affect the outcome.

(iii) Another such example: The vectors

$$v_1 = (-\infty, -\infty, 1), \quad v_2 = (1, 1, -\infty), \quad \text{and} \quad v_3 = (-\infty, 1, 1)$$

are tropically independent, although classical spanning with respect to them (and thus also tropical spanning) is not unique; e.g., $(3, 3, 1) = 2v_2 + v_3 = v_1 + 2v_2$.

(iv) Another such example: Consider the vectors

$$v_1 = (1, 4, 3), \quad v_2 = (2, 3, 4), \quad \text{and} \quad v_3 = (0, 20, 20).$$

Then $(3, 20, 20) = 1v_2 + v_3 = 3v_1 + v_3$.

(v) Another such example: Consider the space V spanned by the five critical vectors

$$(0, -\infty, 0, -\infty, 0, -\infty), \quad (-\infty, 0, -\infty, 0, -\infty, 0),$$

$$(0, -\infty, -\infty, 0, -\infty, -\infty), \quad (-\infty, 0, -\infty, -\infty, 0, -\infty), \quad (-\infty, -\infty, 0, -\infty, -\infty, 0).$$

Then $(0, 0, 0, 0, 0, 0)$ is the sum of the first two vectors as well as the last three.

It does not follow from Lemma 2.24 that for $S = \{w_1, \dots, w_n\}$, there is a ν -maximal set of $\alpha_1, \dots, \alpha_\ell \in \mathcal{T}$ such that $v \models_{\text{gs}} \sum_{i=1}^\ell \alpha_i w_i$. For example, in logarithmic notation take

$$v = (1, 1), \quad w_1 = (1, 0), \quad \text{and} \quad w_2 = (1, 1).$$

Then $v = \alpha w_1 + w_2$ for all $\alpha < 0$, but taking $\alpha = 0$ yields $w_1 + w_2 = (1^\nu, 1)$.

Proposition 5.9. For any subspace V of $F^{(n)}$, the number of elements of any tropically spanning set S of V is at least $\text{rk}(V)$.

Proof. Take a d-base $\{v_1, \dots, v_m\}$ of V , where $m = \text{rk}(V) \leq n$. By [12, Theorem 3.4], the $m \times n$ matrix whose rows are v_1, \dots, v_m has rank m . Taking a nonsingular $m \times m$ submatrix and erasing all the $n - m$ columns not appearing in this submatrix, we may assume that $m = n$ (since we still have a supertropically generating set which we can shrink to a minimal one).

Writing $v_i \models_{\text{gs}} \sum \alpha_{i,j} s_j$ for suitable $s_j \in S$, we see that some matrix whose rows are various s_j is nonsingular, implying that some subset of m vectors of S is tropically independent, and thus $|S| \geq m$. \square

5.3. s-bases. We are ready for another version of base.

Definition 5.10. An **s-base** (for supertropical base) of a supertropical vector space V (over a supertropical semifield F) is a minimal tropical spanning set S , in the sense that no proper subset of S tropically spans V .

As we shall see in Example 5.21 below, a vector space with a finite d-base could still fail to have an s-base. Even when an s-base exists, it could be considerably larger than any d-base.

Example 5.11. Elements of a vector space V may be tropically dependent on a subspace W but not tropically spanned by W , as indicated in Example 5.8(i).

Example 5.12. Let V be the subspace of R^2 spanned by $S = \{(1, 1), (1^\nu, 1), (1, 1^\nu)\}$ in logarithmic notation, equipped with the standard ghost module.

Each of these vectors alone comprises a d-base of V , whereas S is an s-base of V .

Note that an s-base S need not be finite. On the other hand, obviously any finite tropical spanning set contains an s-base, so any finitely spanned vector space has an s-base. In order to coordinate the definitions of s-base and d-base we introduce the following definition.

Definition 5.13. A **d,s-base** is an s-base which is also a d-base. A supertropical vector space V is **finite dimensional** if it has a finite d,s-base.

Proposition 5.14. The cardinality of the s-base S of a finite dimensional vector space V is precisely $\text{rk}(V)$.

Proof. $|S| \geq \text{rk}(V)$ by Proposition 5.9. But we get equality, since by definition S is itself a d-base. \square

Example 5.15. Suppose S is a tropically independent subset of V . Then S is a d,s-base of the subspace of V tropically spanned by S . These are the subspaces of greatest interest to us, and will be studied further, following Definition 6.13.

Example 5.16. There are four possible sorts of nonzero subspaces of $F^{(2)}$ tropically spanned by a set S of tangible elements over a supertropical semifield F , writing $\{\varepsilon_1 = (\mathbb{1}_R, \mathbb{0}_R), \varepsilon_2 = (\mathbb{0}_R, \mathbb{1}_R)\}$ for the standard base:

- (i) The plane $F^{(2)}$ itself.
- (ii) A half-plane – of tangible rank 2, having tangible s-base containing ε_1 or ε_2 , as well as one tangible element $\alpha_1 \varepsilon_1 + \alpha_2 \varepsilon_2$ for $\alpha_1, \alpha_2 \in \mathcal{T}$;
- (iii) A planar strip – of tangible rank 2, having tangible s-base $\{\alpha_1 \varepsilon_1 + \alpha_2 \varepsilon_2, \beta_1 \varepsilon_1 + \beta_2 \varepsilon_2\}$, where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathcal{T}$;
- (iv) A subspace of tangible rank 1, each pair of whose elements are tropically dependent. The tangible vectors are all multiples of a single vector.

One also has examples of non-tangibly generated subspaces of $F^{(2)}$, such as $W = \{(\alpha, \alpha^\nu) : \alpha \in F\}$.

5.4. Critical elements versus s-bases. Since s-bases are involved in the actual generation of the space, they are more in tune with the classical theory of convexity, and can be studied combinatorially. Here is another way to view the s-base, which is inspired by the literature on convex spaces. For convenience, we take R to be a supertropical semifield. We say that two elements v, w in a supertropical vector space V are **projectively equivalent**, written $v \sim w$, iff $v = \alpha w$ for some tangible element $\alpha \in R$. Accordingly, we define the equivalence class of v as

$$[v]_\sim := \{w \in V \mid w \sim v\}.$$

Definition 5.17. A vector v in a supertropical vector space V is **critical** if we cannot write $v \stackrel{\text{gs}}{=} v_1 + v_2$ for $v_1, v_2 \in V \setminus [v]_{\sim}$. Taking one representative for each class $[v]_{\sim}$, a **tropical critical set** of V is defined as a set of representatives of all the critical elements of V .

Critical elements correspond to “extreme points” over the max-plus algebra in [4], who show that every point in $R^{(n)}$ is a linear combination of at most $n+1$ extreme points. There is a basic connection between criticality and almost tangible.

Lemma 5.18. Suppose $v \stackrel{\text{gs}}{=} \alpha v + w$ for $\alpha \in \mathcal{T}$, $v, w \in V$, and $v \notin \mathcal{H}_0$. Then $\alpha \leq_{\nu} e$. Furthermore:

- (1) If $\alpha <_{\nu} e$, then $v \stackrel{\text{gs}}{=} w$.
- (2) Suppose $\alpha \in \mathcal{T}_e$; i.e., $\alpha \cong_{\nu} e$. If $w \in \mathcal{H}_0$, then $v = \alpha v$. For any $w \in V$,

$$v = \alpha^2 v + ew' = \alpha^2 v,$$

where $w' \stackrel{\text{gs}}{=} w$.

Proof. Write $v = \alpha v + w'$, where $w' \stackrel{\text{gs}}{=} w$.

- (1): If $\alpha >_{\nu} e$, then

$$\begin{aligned} v &= \alpha v + w' = (\alpha + \mathbb{1}_F)v + w' \\ &= v + \alpha v + w' = v + v = ev \in \mathcal{H}_0, \end{aligned}$$

a contradiction. Hence, $\alpha \leq_{\nu} e$.

If $\alpha <_{\nu} e$, then $\alpha = \alpha + \mathbb{1}_F$, implying

$$v = (\alpha + \mathbb{1}_F)v = \alpha v + v = \alpha v + \alpha v + w' = e\alpha v + w' \stackrel{\text{gs}}{=} w,$$

proving (1).

- (2): Thus, we assume that $\alpha \in \mathcal{T}_e$. If $w = ew$, then

$$\begin{aligned} v &= \alpha v + w' = \alpha(\alpha v + w') + w' \\ &= \alpha^2 v + (\alpha + \mathbb{1}_F)w' = \alpha^2 v + ew' \\ &= \alpha(\alpha v + ew') = \alpha v. \end{aligned}$$

For any w , if $\alpha \in \mathcal{T}_e$, then

$$v = \alpha v + w' = \alpha(\alpha v + w') + w' = \alpha^2 v + (\alpha + \mathbb{1}_F)w' = \alpha^2 v + ew'.$$

Hence, $v = \alpha^2 v$ by the previous assertion. □

Proposition 5.19. Any critical element $v \in V$ is almost tangible.

Proof. Otherwise $v = w + w'$ for suitable $w \in V$, $w' \in \mathcal{H}_0$, for which $w \neq v$, but by criticality, $w = \alpha v$ for $\alpha \in \mathcal{T}$. First assume that $v \notin \mathcal{H}_0$. Then, by Lemma 5.18, $\alpha \leq_{\nu} e$, and furthermore $\alpha \in \mathcal{T}_e$, since otherwise $v \stackrel{\text{gs}}{=} ew'$ contrary to $v \notin \mathcal{H}_0$. But now, by Lemma 5.18, $v = \alpha v = w$, a contradiction.

Hence we may assume that $v \in \mathcal{H}_0$, and thus

$$w = \alpha v = (\alpha e)v = ev = v,$$

again a contradiction. □

Lemma 5.20. An almost tangible element $v \in V$ is critical iff it is not tropically spanned by $V \setminus [v]_{\sim}$, i.e. $v \not\stackrel{\text{gs}}{=} \sum \alpha_i w_i$ for any $\alpha_i \in \mathcal{T}$, $w_i \in V \setminus v$.

Proof. (\Rightarrow) Suppose on the contrary that $v \stackrel{\text{gs}}{=} \sum_{i=1}^t \alpha_i w_i$; by definition of criticality, $t > 1$. Then taking $v_1 = \alpha_1 w_1$ and $v_2 = \sum_{i=2}^t \alpha_i w_i$, we must have $v_2 \in [v]_{\sim}$, and conclude by induction on t . □

Clearly a tropical critical set of a vector space V is projectively unique, but could be empty.

Example 5.21.

- (i) The standard base $\{\varepsilon_1, \dots, \varepsilon_n\}$ of $R^{(n)}$ is also its tropical critical set.
- (ii) The tropical critical set of the subspace $W = R^{(2)} \setminus ([\varepsilon_1]_{\sim} \cup [\varepsilon_2]_{\sim})$ is empty.
- (iii) $W = R^{(2)} \setminus [\varepsilon_1]_{\sim}$ has the tropical critical set $[\varepsilon_2]_{\sim}$, but has no s -base.

Despite the last two examples, some positive information is available.

Lemma 5.22. *Any tropical spanning set S contains a tropical critical set of V .*

Proof. Suppose $v \in V$ is critical. By hypothesis on S , v is tropically spanned by S but, by Lemma 5.20, it must be an element of S (up to projective equivalence). \square

Theorem 5.23. *Suppose V has an s -base S . Then S is precisely the tropical critical set of V .*

Proof. In view of Lemma 5.22, it remains to show that each element of S is critical. Suppose $v \in S$ is not critical. Then $v = v_1 + v_2$ where $v_1, v_2 \notin \mathcal{T}v$. Thus, when we write

$$v_1 = \sum \alpha_{1,i} s_{1,i} + w_1 \quad \text{and} \quad v_2 = \sum \alpha_{2,i} s_{2,i} + w_2$$

for $\alpha_{1,i}, \alpha_{2,i} \in \mathcal{T}$ and $w_1, w_2 \in \mathcal{H}_0$, we must have v appearing in one of the sums (for otherwise $v = v_1 + v_2$ is tropically spanned by the other elements of S , contrary to hypothesis).

Thus, we may assume $s_{1,1} = v$, and we have

$$v_1 \models_{\text{gs}} \alpha_1 v + \sum_{i \neq 1} \alpha_{1,i} s_{1,i}$$

and similarly $v_2 \models_{\text{gs}} \alpha_2 v + \sum_{i \neq 1} \alpha_{2,i} s_{2,i}$. (Formally, we permit $\alpha_2 = 0_F$.) We also write $v_j = \alpha_j + x_j$ where $x_j \models_{\text{gs}} \sum_{i \neq 1} \alpha_{j,i} s_{j,i}$.

Now

$$v = v_1 + v_2 = \beta v + x,$$

where $\beta = \alpha_{1,1} + \alpha_{2,1}$ and $x = x_1 + x_2$. But then $\beta \leq_{\nu} e$, by Lemma 5.18, which also says that if $\beta <_{\nu} e$, then $v \models_{\text{gs}} x$, contrary to S being an s -base. Thus, we may conclude that $\beta \cong_{\nu} e$. By symmetry, we assume that $\alpha_1 \cong_{\nu} e$. If $\alpha_2 <_{\nu} e$, then $v_2 \models_{\text{gs}} x_2$, and

$$\begin{aligned} v &= v_1 + v_2 = \alpha_1 v + x_1 + v_2 \\ &= \alpha_1(\alpha_1 v + x_1 + v_2) + x_1 + v_2 \\ &= \alpha_1^2 v + (\alpha_1 + 1_F)(x_1 + v_2) \\ &= \alpha_1^2 v + e(x_1 + v_2), \end{aligned}$$

and thus

$$\begin{aligned} v_1 &= \alpha_1^2 v + e(x_1 + v_2) \\ &= \alpha_1(\alpha_1^2 v + e(x_1 + v_2)) + x_1 + v_2 \\ &= \alpha_1^3 v + e(x_1 + v_2) \\ &= \alpha_1(\alpha_1^2 v + e(x_1 + v_2)) = \alpha_1 v. \end{aligned}$$

Thus, we are done for $\alpha_2 <_{\nu} e$, and may assume that $\alpha_2 \in \mathcal{T}_e$. Then

$$v = (\alpha_1 + \alpha_2)v + x_1 + x_2 = ev + x,$$

implying $v = ex \in \mathcal{H}$ and thus $\alpha_j v = \alpha_j ev = ev$ for $j = 1, 2$. Hence

$$v = v_1 + v_2 = \alpha_1 v + x_1 + \alpha_2 v + x_2 = (\alpha_1 + \alpha_2)v + x = ev + x,$$

and thus $v = ev + ex$ by Lemma 5.18, implying

$$\begin{aligned} v_1 &= ev + x_1 = ev + ex + x_1 \\ &= ev + ex_1 + ex_2 + x_1 \\ &= ev + ex_1 + ex_2 = ev + ex = v. \end{aligned}$$

\square

This theorem is generalized in [7].

Corollary 5.24. *The s-base (if it exists) of a supertropical vector space is unique up to multiplication by tangible elements of R , and is comprised of almost tangible elements.*

By Corollary 5.24, we have the following striking result:

Theorem 5.25. *The s-base (if it exists) of a supertropical vector space is unique up to multiplication by scalars.*

Example 5.26. *The only s-bases of the supertropical vector space $V = R^{(n)}$ are its classical bases $S = \{\alpha_1 \varepsilon_1, \dots, \alpha_n \varepsilon_n\}$, where $\alpha_1, \dots, \alpha_n \in \mathcal{T}$.*

One also has the following tie between critical sets and s-bases.

Proposition 5.27. *Any critical set C of a supertropical vector space V is an s-base of the subspace W tropically spanned by C .*

Proof. By hypothesis, C tropically spans W , so we need only check minimality. But for any $v \in C$, by definition, $C \setminus \{v\}$ does not tropically span v . \square

5.5. Thick subspaces.

Definition 5.28. *A subspace W of a supertropical vector space (V, \mathcal{H}_0) is **thick** if $\text{rk}(W) = \text{rk}(V)$.*

For example, the subspace $\alpha V \subseteq V$ is thick, for any $\alpha \in \mathcal{T}$. Likewise, any subspace of $R^{(n)}$ containing n tropically independent vectors is thick.

Remark 5.29. *By definition, any thick subspace of a thick subspace of V is thick in V .*

Remark 5.30. *Any thick subspace W of a supertropical vector space (V, \mathcal{H}_0) contains a d-base of V . Indeed, by definition, for $n = \text{rk}(V)$, W contains a set of n tropically independent elements, which must be a maximal tropically independent set in V , by definition of rank.*

Thus, V is tropically dependent on any thick subspace.

Example 5.31. *There exists an infinite chain of thick subspaces $W_1 \subset W_2 \subset \dots$ of $V = D(\mathbb{R})^{(2)}$, where W_k is the strip tropically spanned by $\{(k, 0), (0, k)\}$, $k \in \mathbb{N}^+$. Thus, $\{(k, 0), (0, k)\}$ is not an s-base of $D(\mathbb{R})^{(2)}$. (One could expand this to an uncountable chain by taking $k \in \mathbb{R}^+$.)*

5.6. Change of base matrices. We write P_π for the permutation matrix whose entry in the $(i, \pi(i))$ position is 1_R (for each $1 \leq i \leq n$) and 0_R elsewhere. Likewise, we write $\text{diag}\{a_1, \dots, a_n\}$ for the diagonal matrix whose entry in the (i, i) position is a_i and 0_R elsewhere, and denote it as D . We call the product $P_\pi D$ of a permutation matrix and a tangible (nonsingular) diagonal matrix, with each diagonal entry $\neq 0_R$, a **generalized permutation matrix**, and denote it as $\tilde{P}_{\pi; D}$.

Recall from [9, Proposition 3.9] that over a supertropical semifield, a matrix is invertible iff it is a generalized permutation matrix $\tilde{P}_{\pi; D}$ with D nonsingular. In particular, the set of all generalized permutation matrices form a group whose unit element is I .

Definition 5.32. *Given an s-base $\mathcal{B} = \{v_1, \dots, v_n\}$ and another s-base $\mathcal{B}' = \{v'_1, \dots, v'_n\}$ of $V \subseteq F^{(n)}$, whose respective row matrices are denoted A and A' , a **change of base matrix** is a matrix P such that*

$$A' = PA; \tag{5.3}$$

Proposition 5.33. *The generalized permutation matrices are the only change of base matrices of s-bases (and thus classical bases).*

Proof. Immediate by Theorem 5.25. \square

Remark 5.34. *It follows from Proposition 5.33, applied to the standard base, that the matrix A is the matrix of a classical base iff A is a generalized permutation matrix.*

Example 5.35. *Any classical base of $R^{(n)}$ (after reordering indices) must be of the form*

$$b_1 = (r_1, 0_R, \dots, 0_R), \quad b_2 = (0_R, r_2, 0_R, \dots, 0_R), \quad \dots, \quad b_n = (0_R, 0_R, \dots, r_n),$$

where $r_i \in \mathcal{T}$ are invertible and tangible.

6. LINEAR TRANSFORMATIONS OF SUPERTROPICAL VECTOR SPACES, AND THE DUAL SPACE

Our main goal in this section is to introduce supertropical linear transformations, and use these to define the dual space with respect to a d,s-base \mathcal{B} , and to show that it has the canonical dual s-base given in Theorem 6.20; this enables us to identify the double dual space with $V_{\mathcal{B}}$. (A version of a dual space for idempotent semimodules, in the sense of dual pairs, leading to a Hahn-Banach type-theorem is given in [3].)

6.1. Supertropical maps. Recall that a **module homomorphism** $\varphi : V \rightarrow V'$ of modules over a semiring R satisfies

$$\varphi(v + w) = \varphi(v) + \varphi(w), \quad \varphi(av) = a\varphi(v), \quad \forall a \in R, v, w \in V.$$

We weaken this a bit in the supertropical theory.

Definition 6.1. Given supertropical vector spaces (V, \mathcal{H}_0) and (V', \mathcal{H}'_0) over a supertropical semifield F , a **supertropical map**

$$\varphi : (V, \mathcal{H}_0) \rightarrow (V', \mathcal{H}'_0)$$

is a function satisfying

$$\varphi(v + w) \models_{\text{gs}} \varphi(v) + \varphi(w), \quad \varphi(\alpha v) = \alpha\varphi(v), \quad \forall \alpha \in F, v, w \in V, \quad (6.1)$$

as well as

$$\varphi(\mathcal{H}_0) \subseteq \mathcal{H}'_0. \quad (6.2)$$

We write $\text{Hom}(V, V')$ for the set of supertropical maps from V to V' , which is viewed as a vector space over F in the usual way. A supertropical map is **strict** if it is a module homomorphism.

The modules over a given semiring with ghosts form a category, whose morphisms are the supertropical maps of modules with ghosts.

Remark 6.2. The second condition of (6.1) implies

$$\varphi(v^\nu) = \varphi(ev) = e\varphi(v) = \varphi(v)^{\nu'};$$

i.e., $\varphi \circ \nu = \nu' \circ \varphi$.

In case $\mathcal{H}_0 = eV$, the standard ghost submodule, (6.2) follows formally from (6.1).

Remark 6.3. One may wonder why we have required $\varphi(\alpha v) = \alpha\varphi(v)$ and not just $\varphi(\alpha v) \models_{\text{gs}} \alpha\varphi(v)$.

In fact, these are equivalent when $\alpha \in \mathcal{T}$, since F is a supertropical semifield. Indeed, assume that $\varphi(\alpha v) \models_{\text{gs}} \alpha\varphi(v)$ for any $\alpha \in \mathcal{T}$ and $v \in V$. Then also $\alpha^{-1} \in \mathcal{T}$. By hypothesis,

$$\alpha^{-1}\varphi(\alpha v) \models_{\text{gs}} \alpha^{-1}\alpha\varphi(v) = \varphi(v)$$

and

$$\varphi(v) = \varphi(\alpha^{-1}\alpha v) \models_{\text{gs}} \alpha^{-1}\varphi(\alpha v),$$

so by antisymmetry, $\alpha^{-1}\varphi(\alpha v) = \varphi(v)$, implying $\varphi(\alpha v) = \alpha\varphi(v)$.

Lemma 6.4. If $v \models_{\text{gs}} w$ then $\varphi(v) \models_{\text{gs}} \varphi(w)$.

Proof. Write $v = w + w'$ where $w' \in \mathcal{H}_0$. Then

$$\varphi(v) \models_{\text{gs}} \varphi(w) + \varphi(w') \models_{\text{gs}} \varphi(w).$$

□

Lemma 6.5. If $v \geq_\nu w$, then $\varphi(v) \geq_\nu \varphi(w)$.

Proof. By definition, $v^\nu \geq w^\nu$, implying $\varphi(v)^\nu = \varphi(v^\nu) \geq_\nu \varphi(w^\nu) = \varphi(w)^\nu$. (Since they are ghosts, $\varphi(v^\nu) \models_{\text{gs}} \varphi(w^\nu)$ is the same as $\varphi(v^\nu) \geq_\nu \varphi(w^\nu)$.) □

Proposition 6.6. *If $V = F$, then any supertropical map $\varphi : V \rightarrow V'$ is strict.*

Proof. We need to show that $\varphi(a+b) = \varphi(a) + \varphi(b)$ for all $a, b \in V$. First assume that $a >_\nu b$. Then $\varphi(a) \geq_\nu \varphi(b)$. But $\varphi(a+b) = \varphi(a)$. If $\varphi(a) >_\nu \varphi(b)$, then $\varphi(a+b) = \varphi(a) + \varphi(b)$. If $\varphi(a) \cong_\nu \varphi(b)$, then

$$\varphi(a) = \varphi(a+b) \models_{\text{gs}} \varphi(a)^\nu,$$

implying $\varphi(a) \in \mathcal{H}'_0$ and $\varphi(a+b) = \varphi(a)^\nu = \varphi(a) + \varphi(b)$.

Thus, we may assume that $a \cong_\nu b$. But then

$$\varphi(a+b) = \varphi(a)^\nu = \varphi(a) + \varphi(b)$$

since $\varphi(a) \cong_\nu \varphi(b)$ by Lemma 6.5. □

Remark 6.7. *There are two advantages that strict supertropical maps have over supertropical maps. First, $(\varphi(V), \varphi(\mathcal{H}_0))$ is a submodule of (V', \mathcal{H}'_0) , for any strict supertropical map $\varphi : V \rightarrow V'$, whereas this may not be so for other supertropical maps.*

Secondly, any strict supertropical map from $F^{(n)} \rightarrow F^{(n)}$ is defined up to ghost surpassing by its action on the standard base. In particular, the strict supertropical map $\varphi : F^{(n)} \rightarrow F^{(n)}$ can be described in terms of $n \times n$ matrices over F . (Proposition 5.33 shows that when these maps are onto, the corresponding matrices are generalized permutation matrices.) Any supertropical map agreeing with φ on the standard base must ghost surpass φ , so in this sense the strict supertropical maps are the “minimal” supertropical maps with respect to ghost surpassing.

Definition 6.8. *Given a supertropical map $\varphi : V \rightarrow V'$ of modules with ghosts, we define the **ghost kernel***

$$\text{g-ker}(\varphi) := \varphi^{-1}(\mathcal{H}'_0) = \{v \in V : \varphi(v) \in \mathcal{H}'_0\},$$

*an R -submodule of V . We say that φ is **ghost monic** if $\varphi^{-1}(\mathcal{H}'_0) = \mathcal{H}_0$.*

Definition 6.9. *A supertropical map $\varphi : V \rightarrow W$ of vector spaces of rank n is called **tropically onto** if $\varphi(V)$ contains a d -base of W of rank n . An **iso** is a supertropical map that is both ghost monic and tropically onto. (Note this need not be an isomorphism in the usual sense, since φ need not be onto.)*

Remark 6.10. *The composition of isos is an iso, in view of Remark 5.29.*

6.1.1. Linear functionals.

Definition 6.11. *Suppose $V = (V, \mathcal{H}_0)$ is a vector space over a supertropical semifield F . The set of supertropical maps*

$$V^* := \text{Hom}(V, F),$$

*is called the **dual F -module** of V , and its elements are called **linear functionals**; i.e., any linear functional $\ell \in V^*$ satisfies*

$$\ell(v_1 + v_2) \models_{\text{gs}} \ell(v_1) + \ell(v_2), \quad \ell(av_1) = a\ell(v_1), \quad \ell(\mathcal{H}_0) \subseteq \mathcal{G}_0$$

for any $v_1, v_2 \in V$ and $a \in F$.

A linear functional $\ell : V \rightarrow F$ is called a **ghost functional** if $\ell(v) \in \mathcal{G}_0$ for all $v \in V$. We write $\mathcal{H}_0^* \subset V^*$ for the subset of all the ghost linear functionals; this is the ghost submodule of V^* . $(V^*, \mathcal{H}_0^*, \nu^*)$ is a supertropical module over F , under the natural operations

$$(\ell_1 + \ell_2)(v) = \ell_1(v) + \ell_2(v), \quad (a\ell)(v) = a\ell(v), \quad \nu^*\ell(v) = \ell(v)^\nu,$$

for $a \in F, v \in V$.

6.2. Linear functionals on subspaces of $F^{(n)}$. The idea here is to develop a theory of linear functionals for n -dimensional subspaces $V \subseteq F^{(n)}$ (Often $V = F^{(n)}$).

Towards this end, we want a definition of linear functionals that respects a given d-base $\mathcal{B} = \{b_1, \dots, b_n\}$ of V . We define the matrix

$$A^{\nabla} := A^{\nabla} A A^{\nabla},$$

cf. [10, Remark 2.14], and recall that $I_A = A A^{\nabla}$ and $I'_A = A^{\nabla} A$ as defined in Equations (3.1) and (3.2). Since the elements of \mathcal{B} are tropically independent, the matrix $A = A(\mathcal{B})$ is nonsingular, and so are the matrices

$$I_A = A A^{\nabla}, \quad A^{\nabla} = A^{\nabla} A A^{\nabla} = A^{\nabla} I_A, \quad \text{and} \quad I'_A = A^{\nabla} A,$$

as well as $I_A A$ (since $I_A A A^{\nabla} = I_A^2 = I_A$ is nonsingular).

Definition 6.12. A d-base \mathcal{B} is **closed** if $I_A \mathcal{B} = \mathcal{B}$.

There is an easy way to get a closed d-base from an arbitrary d-base \mathcal{B} . From now on we set the matrix

$$A := A(\mathcal{B}).$$

Definition 6.13. Write $A_{\mathcal{B}} = I_A A$, and let $\overline{\mathcal{B}}$ denote the rows of $A_{\mathcal{B}}$. Let

$$V_{\mathcal{B}} := \{A_{\mathcal{B}} v : v \in V\},$$

the thick subspace of V spanned by $\overline{\mathcal{B}}$.

$V_{\mathcal{B}}$ is the subspace of interest for us, since it is invariant under the action of the matrix A .

Remark 6.14. $\overline{\mathcal{B}}$ is obviously spanned by \mathcal{B} , but since $I_A A$ is nonsingular, $\overline{\mathcal{B}}$ also is a d-base of V , and clearly $\overline{\mathcal{B}}$ is closed since $I_A^2 = I_A$. Thus, \mathcal{B} is a d,s-base of $V_{\mathcal{B}}$.

The d-base $\overline{\mathcal{B}}$ is easier to compute with, since now we have

$$I_{A_{\mathcal{B}}} A_{\mathcal{B}} = A_{\mathcal{B}}.$$

From now on, replacing \mathcal{B} by $\overline{\mathcal{B}}$ if necessary, we assume that the d-base \mathcal{B} of V is closed.

Rather than dualizing all of V , we turn to the space

$$V_{\mathcal{B}}^* := \text{Hom}(V_{\mathcal{B}}, F).$$

Define $L_A \in \text{Hom}(V, V)$ by

$$L_A(v) := A^{\nabla} v.$$

We also define the map $\tilde{L}_A : V \rightarrow V$ by

$$\tilde{L}_A(v) := I_A v.$$

Remark 6.15. $(\tilde{L}_A)^2 = \tilde{L}_A$, and \tilde{L}_A is the identity on $V_{\mathcal{B}}$ since

$$I_A(I_A A v) = I_A^2 A v = I_A A v.$$

Likewise, $L_A(v) = A^{\nabla} v$ for all $v \in V_{\mathcal{B}}$.

Lemma 6.16. If $\ell \in V_{\mathcal{B}}^*$, then $\ell = (\ell \circ \tilde{L}_A)|_{V_{\mathcal{B}}}$ on $V_{\mathcal{B}}$. In other words,

$$V_{\mathcal{B}}^* = \{(\ell \circ \tilde{L}_A)|_{V_{\mathcal{B}}} : \ell \in V^*\}.$$

Proof. Follows at once from the remark. □

Lemma 6.17. $V_{\mathcal{B}}^*$ is a supertropical vector space, whose ghost submodule $\mathcal{H}_0(V_{\mathcal{B}}^*)$ is $\{f|_{V_{\mathcal{B}}} : f \in \mathcal{H}_0^*\}$.

Proof. Suppose $f' \in \mathcal{H}_0(V_{\mathcal{B}}^*)$. Let $f = f' \circ \tilde{L}_A \in \mathcal{H}_0^*$. Then $f' = f|_{V_{\mathcal{B}}}$. The other inclusion is obvious. □

Definition 6.18. Given a (closed) d-base $\mathcal{B} = \{b_1, \dots, b_n\}$ of V , define $\epsilon_i : V_{\mathcal{B}} \rightarrow F$ by

$$\epsilon_i(v) = b_i^{\text{t}} L_A(v),$$

the scalar product of b_i and $A^{\nabla} v$. Also, define $\mathcal{B}^* = \{\epsilon_i : 1 \leq i \leq n\}$.

When v is tangible, we saw in Remark 4.19 that

$$v \underset{gd}{\gamma} \sum_{i=1}^n b_i \widehat{\epsilon_i(v)}$$

is a saturated tropical dependence relation of v on the b_i 's; this is the motivation behind our definition.

Remark 6.19.

- (i) ϵ_i is a linear functional. Also, by definition, $\epsilon_i(b_j)$ is the i, j position of $AA^\nabla = I_A$, a quasi-identity, which implies

$$\epsilon_i(b_i) = 1_R; \quad \epsilon_i(b_j) \in \mathcal{G}, \quad \forall i \neq j.$$

Hence,

$$\sum_{i=1}^n \alpha_i \epsilon_i(b_j) \underset{gs}{=} \alpha_j \epsilon_j(b_j) = \alpha_j.$$

$$(ii) \sum b_i \epsilon_i(v) = A \begin{pmatrix} \epsilon_1(v) \\ \vdots \\ \epsilon_n(v) \end{pmatrix} = AA^\nabla v = v, \text{ for } v \in V_{\mathcal{B}}.$$

Theorem 6.20. If R is a supertropical semifield and \mathcal{B} is a closed s -base of V , then $\{\epsilon_i : 1 \leq i \leq n\}$ is a closed s -base of $V_{\mathcal{B}}^*$.

Proof. For any $\ell \in V_{\mathcal{B}}^*$, we write $\alpha_i = \ell(b_i)$, and then see from Remark 6.19 that $\sum \alpha_i \epsilon_i \underset{gs}{=} \ell$ on $V_{\mathcal{B}}$.

It remains to show that the $\{\epsilon_i : i = 1, \dots, n\}$ are tropically independent. If $\sum_{i=1}^n \beta_i \epsilon_i$ were ghost for some $\beta_i \in \mathcal{T}_0$, we would have $\sum \beta_i b_i^t A^\nabla$ ghost. Let D denote the diagonal matrix $\{\beta_1, \dots, \beta_n\}$, and let $\mathcal{I} = \{i : \beta_i \neq 0_R\}$, and assume there are k such tangible coefficients β_i . Then for any $i \notin \mathcal{I}$ we have $\beta_i = 0$, implying the i row of the matrix DI_A is zero. But the sum of the rows of the matrix DI_A corresponding to indices from \mathcal{I} would be $\sum \beta_i b_i^t A^\nabla$, which is ghost, implying that these k rows of DI_A are dependent; hence DI_A has rank $\leq k-1$. On the other hand, the k rows of DI_A corresponding to indices from \mathcal{I} yield a $k \times k$ submatrix of determinant $\prod_{i \in \mathcal{I}} \beta_i \in \mathcal{T}$, implying its rank $\geq k$ by [12, Theorem 3.4], a contradiction. \square

In the view of the theorem, we denote $\mathcal{B}^* = \{\epsilon_i : 1 \leq i \leq n\}$, and call it the (tropical) **dual s-base** of \mathcal{B} .

Write $V_{\mathcal{B}}^{**}$ for $(V_{\mathcal{B}}^*)^*$. Define a map

$$\Phi : V_{\mathcal{B}} \rightarrow V_{\mathcal{B}}^{**},$$

given by $v \mapsto f_v$, where

$$f_v(\ell) = \ell(v).$$

Remark 6.21. Let v_j denote the j -th row of A^∇ , i.e., $v_j = b_j'$. Since $AA^\nabla = I_A$ is a quasi-identity matrix, we see that

$$f_{b_j}(\epsilon_i) = \epsilon_i(b_j) = b_i^t A^\nabla b_j = \begin{cases} \frac{|A|}{|A|} b_i = b_i, & i = j; \\ \text{ghost}, & i \neq j. \end{cases}$$

Lemma 6.22. Suppose $v = \sum \alpha_i b_i$, for $\alpha_i \in \mathcal{T}$. Then $f_v(\epsilon_i) \notin \mathcal{H}_0$ for some i .

Proof. (\Leftarrow) The assertion is obvious. (\Rightarrow) Suppose $\epsilon_i(v) = f_v(\epsilon_i) \in \mathcal{H}_0$ for each i . Then $\sum \alpha_i b_i \in \mathcal{H}_0$, contrary to the b_i being tropically independent. \square

Example 6.23. Suppose $V = F^{(n)}$, a supertropical vector space. The map $\Phi : V \rightarrow V^{**}$ is a vector space isomorphism when \mathcal{B} is the standard base.

Proposition 6.24. For any $v \in V$, define $v^{**} \in V^{**}$ by $v^{**}(\ell) = \ell(v)$. The map $\Phi : V_{\mathcal{B}} \rightarrow V_{\mathcal{B}}^{**}$ given by $v \mapsto v^{**}$ is an iso of supertropical vector spaces.

Proof. $\Phi(\mathcal{B})$ is a d-base of n elements, which is ghost injective, since any non-ghost vector $v = \sum \alpha_i b_i$ of $\Phi(\mathcal{B})$ has some tangible coefficient α_i , and then $v^{**}(\epsilon_i) = \alpha_i \in \mathcal{T}$. But by Example 6.23, taking the standard classical base, we see that V^{**} has rank n . Hence any supertropical subspace having n tropically independent elements is thick. \square

7. SUPERTROPICAL BILINEAR FORMS

The classical way to study orthogonality in vector spaces is by means of bilinear forms. In this section, we introduce the supertropical analog, providing some of the basic properties. Although the tropical literature deals with orthogonality in terms of the inner product, as described in [1, § 25.6], the supertropical theory leads to a more axiomatic approach.

The notion of supertropical bilinear form follows the classical algebraic theory, although, as to be expected, there are a few surprises, mostly because of the characteristic 2 nature of the theory [6]. In this section, we assume that V is a vector space over a supertropical semifield F .

7.1. Supertropical bilinear forms.

Definition 7.1. A (supertropical) **bilinear form** on supertropical vector spaces $V = (V, \mathcal{H}_0)$ and $V' = (V', \mathcal{H}'_0)$ is a function $B : V \times V' \rightarrow F$ that is a linear functional in each variable; i.e., writing $(v, w) \mapsto B(v, w)$ for $v \in V$ and $w \in V'$, any given u in V and $u' \in V'$, we have linear functionals

$$B(u, \underline{\quad}) w \mapsto B(u, w), \quad B(\underline{\quad}, u') v \mapsto B(v, u'),$$

satisfying $B(V, \mathcal{H}'_0) \subseteq \mathcal{G}_0$ and $B(\mathcal{H}_0, V') \subseteq \mathcal{G}_0$. Thus,

$$B(v_1 + v_2, w_1 + w_2) \stackrel{\text{gs}}{=} B(v_1, w_1) + B(v_1, w_2) + B(v_2, w_1) + B(v_2, w_2),$$

$$B(\alpha v, w) = \alpha B(v, w) = B(v, \alpha w),$$

for all $\alpha \in F$ and $v_i \in V$, and $w_j \in V'$.

When $V' = V$, we say that B is a (**supertropical**) **bilinear form** on the vector space V . We say that a bilinear form B is **strict** if

$$B(\alpha_1 v_1 + \alpha_2 v_2, \beta_1 w_1 + \beta_2 w_2) = \alpha_1 \beta_1 B(v_1, w_1) + \alpha_1 \beta_2 B(v_1, w_2) + \alpha_2 \beta_1 B(v_2, w_1) + \alpha_2 \beta_2 B(v_2, w_2),$$

for all $v_i \in V$ and $w_i \in V'$.

We often suppress B in the notation, writing $\langle v, w \rangle$ for $B(v, w)$. Perhaps surprisingly, one can lift many of the classical theorems about bilinear forms to the supertropical setting, without requiring strictness.

Example 7.2. There is a natural bilinear form $B : V \times V^* \rightarrow F$, given by $B(v, f) = f(v)$, for $v \in V$ and $f \in V^*$.

Remark 7.3.

- (i) There is a natural map $\Phi : V' \rightarrow V^*$, given by $w \mapsto \langle \underline{\quad}, w \rangle$. Likewise, there is a natural map $\Phi : V \rightarrow (V')^*$, given by $v \mapsto \langle \underline{\quad}, v \rangle$.
- (ii) For any bilinear form B , if $v \stackrel{\text{gs}}{=} \sum \alpha_i v_i$ and $w \stackrel{\text{gs}}{=} \sum \beta_j w_j$, for $\alpha_i, \beta_j \in \mathcal{T}$, then

$$\langle v, w \rangle \stackrel{\text{gs}}{=} \sum_{i,j} \alpha_i \beta_j \langle v_i, w_j \rangle. \quad (7.1)$$

For the remainder of this section, we take $V' = V \subseteq F^{(n)}$, a vector space over the supertropical semifield F , and consider a (supertropical) bilinear form B on V .

Definition 7.4. The **Gram matrix** of vectors $v_1, \dots, v_k \in V = F^{(n)}$ is defined as the $k \times k$ matrix

$$\tilde{G}(v_1, \dots, v_k) = \begin{pmatrix} \langle v_1, v_1 \rangle & \langle v_1, v_2 \rangle & \cdots & \langle v_1, v_k \rangle \\ \langle v_2, v_1 \rangle & \langle v_2, v_2 \rangle & \cdots & \langle v_2, v_k \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle v_k, v_1 \rangle & \langle v_k, v_2 \rangle & \cdots & \langle v_k, v_k \rangle \end{pmatrix}. \quad (7.2)$$

The set $\{v_1, \dots, v_k\}$ is **nonsingular** (with respect to B) iff its Gram matrix is nonsingular (see §3).

The **Gram matrix** of V is the Gram matrix of an s -base of V .

Example 7.5. *The quasi-identity*

$$\tilde{G}(v_1, v_2) = \begin{pmatrix} 0 & 1^\nu \\ -\infty & 0 \end{pmatrix} \quad (7.3)$$

(in logarithmic notation) is the Gram matrix of a bilinear form. Note that

$$\langle v_1, v_2 \rangle = 1^\nu > 0^\nu = \langle v_1, v_1 \rangle + \langle v_2, v_2 \rangle.$$

In particular, we have the matrix $\tilde{G} = \tilde{G}(b_1 \dots, b_k)$, which can be written as $(g_{i,j})$ where $g_{i,j} = \langle b_i, b_j \rangle$; (7.1) written in matrix notation becomes

$$\langle v, w \rangle \underset{\text{gs}}{=} v^t \tilde{G} w. \quad (7.4)$$

Of course, the matrix \tilde{G} depends on the choice of tangible s-base \mathcal{B} of V , but this is unique up to multiplication by scalars and permutation, so \tilde{G} is unique up to $P\tilde{G}P^t$ where P is a generalized permutation matrix. In particular, whether or not \tilde{G} is nonsingular does not depend on the choice of s-base.

7.2. Ghost orthogonality. Bilinear forms play a key role in geometry since they permit us to define orthogonality of supertropical vectors. However, as we shall see, orthogonality is rather delicate in this setup.

Definition 7.6. We write $v \perp\!\!\!\perp w$ when $\langle v, w \rangle \in \mathcal{G}_0$, that is $\langle w_1, w_2 \rangle \underset{\text{gs}}{=} 0_F$ (cf. Remark 2.6), and say that v and w are **ghost orthogonal**, or **g-orthogonal** for short. Likewise, subspaces W_1, W_2 of V are **g-orthogonal** if $\langle w_1, w_2 \rangle \in \mathcal{G}_0$ for all $w_i \in W_i$.

A subset S of V is **g-orthogonal** (with respect to a given bilinear form) if any pair of distinct vectors from S is g-orthogonal. The **(left) orthogonal ghost complement** of S is defined as

$$S^\perp = \{v \in V : \langle v, S \rangle \in \mathcal{G}_0\}.$$

The orthogonal ghost complement S^\perp of any set $S \subset V$ is a subspace of V , and $\mathcal{H}_0 \subseteq S^\perp$ for any $S \subset V$. Note that g-orthogonality is not necessarily a symmetric relation.

Definition 7.7. A subspace W of V is called **nondegenerate** (with respect to B), if $W^\perp \cap W \subseteq \mathcal{H}_0$. The bilinear form B is **nondegenerate** if the space V is nondegenerate.

The **radical**, $\text{rad}(V)$, with respect to a given bilinear form B , is defined as V^\perp . Vectors w_i are **radically dependent** if $\sum \alpha_i w_i \in \text{rad}(V)$ for suitable $\alpha_i \in \mathcal{T}_0$, not all 0_F .

Clearly, $\mathcal{H}_0 \subseteq \text{rad}(V)$.

Remark 7.8.

- (i) $\text{rad}(V) = \mathcal{H}_0$ when V is nondegenerate, in which case radical dependence is the same as tropical dependence.
- (ii) Any ghost complement V' of $\text{rad}(V)$ is obviously tropically g-orthogonal to $\text{rad}(V)$, and nondegenerate since

$$\text{rad}(V') \subseteq V' \cap \text{rad}(V) \subseteq \mathcal{H}_0.$$

This observation enables us to reduce many proofs to nondegenerate subspaces, especially when a Gram-Schmidt procedure is applicable (to be described in [6]).

Lemma 7.9. Suppose $\{w_1, \dots, w_m\}$ tropically span a subspace W of V . If $\sum \beta_i \langle v, w_i \rangle \in \mathcal{G}_0$ for each $v \in V$, then $\sum_{i=1}^m \beta_i w_i \in W^\perp$.

Proof. $\langle v, \sum_i \beta_i w_i \rangle \underset{\text{gs}}{=} \sum_i \langle v, \beta_i w_i \rangle = \sum_i \beta_i \langle v, w_i \rangle \in \mathcal{G}_0$ for all $v \in W$. Thus, $\sum_i \beta_i w_i \in W^\perp$. \square

Theorem 7.10. Assume that vectors $w_1, \dots, w_k \in V$ span a nondegenerate subspace W of V . If $|\tilde{G}(w_1, \dots, w_k)| \in \mathcal{G}_0$, then w_1, \dots, w_k are tropically dependent.

Proof. Write $\tilde{G} = \tilde{G}(v_1, \dots, v_k)$. By [9, Theorem 6.6], $|\tilde{G}| \in \mathcal{G}_0$ iff the rows of \tilde{G} are tropically dependent. By the lemma, if $|\tilde{G}| \in \mathcal{G}_0$, then some linear combination of the v_i is in W^\perp . When W is nondegenerate, this latter assertion is the same as saying that the v_i are tropically dependent. \square

Corollary 7.11. *If the bilinear form B is nondegenerate on a vector space V , then the Gram matrix (with respect to any given supertropical d, s -base of V) is nonsingular.*

Remark 7.12. *In case the bilinear form B is strict, we can strengthen Lemma 7.9 to obtain:*

$$\sum_{i=1}^m \beta_i w_i \in W^\perp \quad \text{iff} \quad \sum \beta_i \langle v, w_i \rangle \in \mathcal{G}_0$$

for each $v \in V$. (Indeed, if $\sum_i \beta_i w_i \in W^\perp$, then $\sum_i \beta_i \langle v, w_i \rangle = \langle v, \sum_i \beta_i w_i \rangle \in \mathcal{G}_0$ for all i .)

In this case, we can also strengthen Corollary 7.11 to read:

Corollary 7.13. *A strict bilinear form B is nondegenerate on a supertropical vector space V iff the Gram matrix (with respect to any given supertropical d, s -base of V) is nonsingular.*

7.3. Symmetry of g-orthogonality. In this subsection, we prove the supertropical version of a classical theorem of Artin, that any bilinear form in which g-orthogonality is symmetric must be either an alternate or symmetric bilinear form. In characteristic 2, any alternate form is symmetric, so we would expect our supertropical forms to be symmetric in some sense.

Definition 7.14. *The (supertropical) bilinear form B is **orthogonal-symmetric** if it satisfies the property for all $v_i, w \in V$:*

$$\sum_i \langle v_i, w \rangle \in \mathcal{G}_0 \quad \text{iff} \quad \sum_i \langle w, v_i \rangle \in \mathcal{G}_0, \quad (7.5)$$

for any finite sum taken over $v_i \in V$.

B is **supertropically symmetric** if B is orthogonal-symmetric and satisfies the additional property that $\langle v, w \rangle \cong_\nu \langle w, v \rangle$ for all $v, w \in V$ satisfying $\langle v, w \rangle \in \mathcal{T}$.

A vector $v \in V$ is **isotropic** if $\langle v, v \rangle \in \mathcal{G}_0$; the vector v is **strictly isotropic** if $\langle v, v \rangle = 0_F$.

Remark 7.15. *If every $v \in V$ is strictly isotropic, then the (supertropical) bilinear form B is trivial. (Indeed,*

$$0_F = \langle v + w, v + w \rangle = \langle v, w \rangle + \langle w, v \rangle$$

for all $v, w \in V$, implying $\langle v, w \rangle = \langle w, v \rangle = 0_F$.)

Remark 7.16. *When the bilinear form B is strict, Condition (7.5) reduces to the condition*

$$\langle v, w \rangle \in \mathcal{G}_0 \quad \text{iff} \quad \langle w, v \rangle \in \mathcal{G}_0$$

since, taking $v = \sum_i v_i$, we have

$$\sum_i \langle v_i, w \rangle = \langle v, w \rangle; \quad \langle w, v \rangle = \sum_i \langle w, v_i \rangle.$$

In general, we need Condition (7.5) to carry through the proof of Theorem 7.20 below.

Lemma 7.17. *An orthogonal-symmetric bilinear form B is supertropically symmetric if it satisfies the condition that $\langle v, w \rangle + \langle w, v \rangle \in \mathcal{G}_0$ for all vectors $v, w \in V$.*

Proof. If $\langle v, w \rangle \in \mathcal{G}_0$, then $\langle w, v \rangle \in \mathcal{G}_0$ by orthogonal-symmetry. Thus, we may assume that $\langle v, w \rangle \in \mathcal{T}$. But then $\langle w, v \rangle \in \mathcal{T}$ by orthogonal-symmetry; by hypothesis, $\langle v, w \rangle + \langle w, v \rangle \in \mathcal{G}_0$, implying $\langle v, w \rangle \cong_\nu \langle w, v \rangle$, as desired. \square

Also, the symmetry condition extends to sums.

Lemma 7.18. *If B is supertropically symmetric, then*

$$\sum_i \langle v_i, w \rangle \in \mathcal{T} \quad \text{iff} \quad \sum_i \langle w, v_i \rangle \in \mathcal{T}$$

with $\sum_i \langle v_i, w \rangle = \sum_i \langle w, v_i \rangle \in \mathcal{T}$.

Proof. We may assume that $\sum_i \langle v_i, w \rangle, \sum_i \langle w, v_i \rangle \in \mathcal{T}$, since there is nothing to check if one (and thus the other) is ghost. Take i_1 such that $\langle v_{i_1}, w \rangle$ is the dominant summand of $\sum_i \langle v_i, w \rangle$, and thus is tangible. Likewise, take i_2 such that $\langle w, v_{i_2} \rangle$ is the dominant summand of $\sum_i \langle w, v_i \rangle$, and thus is tangible. By hypothesis $\langle v_{i_1}, w \rangle = \langle w, v_{i_1} \rangle$ and $\langle w, v_{i_2} \rangle = \langle v_{i_2}, w \rangle$. Since these dominate their respective sums, we get $\sum_i \langle v_i, w \rangle = \sum_i \langle w, v_i \rangle \in \mathcal{T}$. \square

We aim to prove that an orthogonal-symmetric (supertropical) bilinear form is supertropically symmetric.

Another important property to check is when $\langle v, w \rangle + \langle w, v \rangle \in \mathcal{G}_0$. This condition means that v is orthogonal to w with respect to the new bilinear form given by $\langle v, w \rangle' := \langle v, w \rangle + \langle w, v \rangle$, and arises here in several assertions.

Lemma 7.19. *Suppose that B is an orthogonal-symmetric bilinear form and $v, w \in V$. Then either $\langle v, w \rangle + \langle w, v \rangle \in \mathcal{G}_0$, or v and w are strictly isotropic.*

Proof. One may assume that $\langle v, w \rangle \in \mathcal{T}$; hence $\langle w, v \rangle \in \mathcal{T}$. If $\langle v, w \rangle \cong_\nu \langle w, v \rangle$ then $\langle v, w \rangle + \langle w, v \rangle \in \mathcal{G}_0$, so we may assume by symmetry that $\langle v, w \rangle >_\nu \langle w, v \rangle$.

First assume that w is nonisotropic. Then $\gamma \langle v, w \rangle + \langle w, w \rangle$ is ghost for $\gamma = \frac{\langle w, w \rangle}{\langle v, w \rangle}$ and tangible for any other tangible γ in F . But $\gamma \langle w, v \rangle + \langle w, w \rangle$ is ghost for $\gamma = \frac{\langle w, w \rangle}{\langle w, v \rangle}$, contradicting orthogonal-symmetry unless $\langle v, w \rangle \cong_\nu \langle w, v \rangle$, implying $\langle v, w \rangle + \langle w, v \rangle \in \mathcal{G}_0$.

Next assume that w is isotropic but $\langle w, w \rangle = \alpha^\nu \neq 0_F$ for $\alpha \in \mathcal{T}$. Then for tangible $\gamma >_\nu \frac{\langle w, w \rangle}{\langle v, w \rangle}$ we see that $\langle \gamma v, w \rangle + \langle w, w \rangle$ is tangible, so $\langle w, \gamma v \rangle + \langle w, w \rangle$ must also be tangible, which is false if $\gamma <_\nu \frac{\langle w, w \rangle}{\langle w, v \rangle}$. This yields a contradiction if $\langle w, v \rangle <_\nu \langle v, w \rangle$, and similarly we have a contradiction if $\langle w, v \rangle >_\nu \langle v, w \rangle$; hence $\langle w, v \rangle \cong_\nu \langle v, w \rangle$, implying $\langle v, w \rangle + \langle w, v \rangle \in \mathcal{G}_0$.

Thus, we may assume that $\langle w, w \rangle = 0_F$. Likewise, $\langle v, v \rangle = 0_F$, since otherwise we would conclude by interchanging v and w . \square

We conclude with our supertropical version of Artin's theorem.

Theorem 7.20. *Every orthogonal-symmetric bilinear form B on a supertropical vector space V is supertropically symmetric.*

Proof. We are done by Lemma 7.19 unless there are vectors $v, w \in V$ for which $\langle v, v \rangle = \langle w, w \rangle = 0_F$ and $\langle v, w \rangle + \langle w, v \rangle \in \mathcal{T}$.

$\alpha := \langle v, w \rangle \in \mathcal{T}$; then $\beta := \langle w, v \rangle \in \mathcal{T}$, and $\alpha + \beta \in \mathcal{T}$. Observe that, if $v' \in V$ such that $\langle v', w \rangle \cong_\nu \alpha$, then $\langle w, v' \rangle \cong_\nu \beta$. Indeed, $\langle v, w \rangle + \langle v', w \rangle = \alpha^\nu$, implying $\langle w, v \rangle + \langle w, v' \rangle \in \mathcal{G}$. But $\langle w, v' \rangle \in \mathcal{T}$, so we conclude that $\langle w, v' \rangle \cong_\nu \beta$.

Now let vector v' be any vector of V . Then

$$\langle v + v', w \rangle \underset{\text{gs}}{=} \langle v, w \rangle + \langle v', w \rangle \neq 0_F.$$

Thus, $\langle v + v', w \rangle \cong_\nu \gamma$, for some $\gamma \in \mathcal{T}$. Let $v'' := \frac{\alpha}{\gamma}(v + v')$. Then

$$\langle v'', w \rangle = \frac{\alpha}{\gamma} \langle v + v', w \rangle \cong_\nu \alpha,$$

and thus $\langle w, v'' \rangle \cong_\nu \beta$, as just observed. Hence, $\langle v'', w \rangle + \langle w, v'' \rangle \notin \mathcal{G}$. Now Lemma 7.19 yields $\langle v'', v'' \rangle = 0_F$. From

$$0_F = \langle \gamma v'', \gamma v'' \rangle \underset{\text{gs}}{=} \langle v, v \rangle + \langle v, v' \rangle + \langle v', v \rangle + \langle v', v' \rangle,$$

we conclude that $\langle v', v' \rangle = 0_F$ for all $v' \in V$; i.e., B is trivial, by Remark 7.15, which is absurd since $\alpha = \langle v, w \rangle \neq 0_F$. Thus, B must be supertropically symmetric. \square

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DEPARTMENT OF MATHEMATICS, BAR-ILAN UNIVERSITY, RAMAT-GAN 52900, ISRAEL
E-mail address: `zzur@math.biu.ac.il`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF REGENSBURG, REGENSBURG, GERMANY
E-mail address: `manfred.knebusch@mathematik.uni-regensburg.de`

DEPARTMENT OF MATHEMATICS, BAR-ILAN UNIVERSITY, RAMAT-GAN 52900, ISRAEL
E-mail address: `rowen@macs.biu.ac.il`